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Jens Høyrup:

Application of Quasi-algebraicTechniques to Geometry



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Chapter 5 Application of Quasi-algebraic Techniques to Geometry

We still have not decided what is to be meant by "algebra." Any distinction between Old Babylonian "algebra" and "quasi-algebra" must therefore remain preliminary—a hypothesis that will allow us to collect the observations that in the end will serve in a more systematic discussion.

Be that as it may, all problems dealt with in Chapters 2–4 can be translated into modern algebraic symbols (albeit with a certain loss of information). On the whole the same can be said about the methods used to resolve them.

Such a translation will not be possible in the problems that are analyzed in the present chapter. There is, however, a fairly close connection between the methods that are applied here and those which we know from the preceding chapters. In this sense at least it seems legitimate to speak of them as "quasi-algebraic."

VAT 8512

Obv.

- 1. A triangle. 30 the width. In the inside two plots,
- 2. the upper surface over the lower surface, 7' went beyond.
- 3. The lower descendant over the upper descendant, 20 went beyond.
- 4. The descendants and the bar what?
- 5. And the surfaces of the two plots what?
- 6. You, 30 the width posit, 7' which the upper surface over the lower surface went beyond posit,
- 7. and 20 which the lower descendant over the upper descendant went beyond posit.
- 8. IGI 20 which the lower descendant over the upper descendant went beyond
- 9. detach: 3' to 7' which the upper surface over the lower surface went beyond
- 10. raise, 21 may your head hold!
- 11. 21 to 30 the width join: 51
- 12. together with 51 make hold: 43'21
- 13. 21 which your head holds together with 21
- 14. make hold: 7'21 to 43'21 join: 50'42.

- 15. 50'42 to two break: 25'21.
- 16. The equal of 25'21 what? 39.
- 17. From 39, 21 the made-hold tear out, 18.
- 18. 18 which you have left is the bar.
- 19. Well, if 18 is the bar,
- 20. the descendants and the surfaces of the two plots what?
- 21. You, 21 which together with itself you have made hold, from 51
- 22. tear out: 30 you leave. 30 which you have left
- 23. to two break, 15 to 30 which you have left raise,
- 24. 7'30 may your head hold!

Edge

- 1. 18 the bar together with 18 make hold:
- 2. 5'24 from 7'30 which your head holds
- 3. tear out: 2'6 you leave.

Rev.

- 1. What to 2'6 may I posit
- 2. which 7' which the upper surface over the lower surface went beyond gives me?
- 3. 3°20' posit. 3°20' to 2'6 raise, 7' it gives you.
- 4. 30 the width over 18 the bar what goes beyond? 12 it goes beyond.
- 5. 12 to $3^{\circ}20'$ which you have posited raise, 40.
- 6. 40 the upper descendant.
- 7. Well, if 40 is the upper descendant,
- 8. the upper surface is what? You, 30 the width,
- 9. 18 the bar heap: 48 to two break: 24.
- 10. 24 to 40 the upper descendant raise, 16'.
- 11. 16' the upper surface. Well, if 16' the upper surface,
- 12. the lower descendant and the lower surface what?
- 13. You, 40 the upper descendant to 20 which the lower descendant over the upper descendant goes beyond
- 14. join, 1' the lower descendant.
- 15. 18 the bar to two break: 9
- 16. to 1' the lower descendant raise, 9'.
- 17. 9' the lower surface.

Many Old Babylonian mathematical problems deal with the partition of fields. The mathematical substance may vary—sometimes the shape of the field is irrelevant and only the area is given together with the specific conditions for

its division; sometimes, as here, what is asked for is a division of a particular geometric shape.

Already before 2200 BCE, Mesopotamian surveyors knew how to divide *a trapezium* into two equal parts by means of a parallel transversal; we shall return in a short while to how they did it. A similar division of a triangle cannot be made exactly without the use of irrational numbers—which means that it could not be done by the Old Babylonian calculators (except with approximation, which was not among the normal teaching aims).

The present problem deals with a variant of the triangle division which *can* be performed exactly. As we see in lines Obv. 1–3 and as shown in Figure 5.1, a triangular field is divided into two parcels (an "upper surface" and a "lower surface") by a "bar," that is, a parallel transversal. For simplicity we may assume the triangle to be rectangular. It is almost certain that the author of the text did as much, and that the "descendants" are thus part of the side; but if we interpret the "descendants" as heights, the calculations are valid for an oblique triangle too.



Figure 5.1: The triangular divided field of VAT 8512, with the auxiliary rectangle.

The two parcels are thus unequal in area. However, we know the difference between their areas, as well as the difference between the appurtenant "descendants." The solution makes use of an unsuspected and elegant ruse and may therefore be difficult to follow.

Lines Obv. 8–10 "raise" the IGI of the difference between the two "descendants" to the difference between the two "surfaces." This means that the text finds the width of a rectangle whose length corresponds to the difference between the partial heights and whose area equals the difference between the partial areas. This width (which is 21) is first memorized and then "joined" to the width of the triangle.

The outcome is a triangle with an attached rectangle—all in all the trapezium shown in Figure 5.1. When prolonging the bar, producing a parallel transversal of the trapezium, we discover that it divides the trapezium into two equal parts—and that is the problem the surveyors had known to solve for half a millennium or more.

Lines Obv. 11–16 show how they had done it: the square on the bisecting transversal is determined as the average between the squares of the parallel sides. The operations that are used ("making hold" and "breaking") show that the process is really thought in terms of geometric squares and average. Figure (5.2) shows why the procedure leads to the correct result. By definition, the average is equidistant from the two extremes. Therefore the gnomon between 21 and 39 must equal that between 39 and 51 ($39^2 - 21^2 = 51^2 - 39^2$); half of these gnomons—the two parts of the shaded trapezium—must therefore also be equal. In the first instance this only concerns a trapezium cut out along the diagonal of a square, but we may imagine the square drawn long (into a rectangle) and perhaps twisted into a parallelogram; none of these operations changes the ratio between areas or parallel linear extensions, and they allow the creation of an arbitrary trapezium. This trapezium will still be bisected, and the sum of the squares on the parallel sides will still be twice that of the parallel transversal.



Figure 5.2: The bisection of the trapezium of BM 8512.

We may take note that the operation of "drawing long" is the same as that change of scale in one direction which we have encountered in the solution of non-normalized problems, and which was also used in TMS XIII, the oil trade (see page 70); we shall meet it again in a moment in the present problem.

Possibly the rule was first found on the basis of concentric squares (see Figure 5.3)—the geometric configuration represented by two or several concentrically nested squares was much appreciated in Babylonian mathematics and may have been so already in the third millennium (as it remained popular among master builders until the Renaissance); the principle of the argument evidently remains the same.



Figure 5.3: The bisection of the trapezium explained by concentric squares.

Line Obv. 17 thus finds the bisecting transversal; it turns out to be 39, and the "bar" between the two original parcels must therefore be 39 - 21 = 18.

The next steps may seem strange. Lines Obv. 21-22 appear to calculate the width of the triangle, but this was one of the given magnitudes of the problem. This means no doubt that we have effectively left behind Figure 5.1, and that the argument is now based on something like Figure 5.2. When we eliminate the additional width 21 we are left with a triangle that corresponds to the initial triangle but which is isosceles—see Figure 5.4.

In order to find the "upper descendant" the text makes the false position that the shortened and isosceles triangle is the one we are looking for. Its length (the sum of the "descendants") is then equal to the width, that is, to 30. In order to find the true triangle we will have to change the scale in the direction of the length.

Lines Obv. 23–24 calculate that the area of the false triangle is 7'30. The two areas in white are equal, and their sum must be $2 \cdot (\frac{1}{2} \cdot (18 \cdot 18)) = 5'24$. The shaded area—which corresponds to the difference between the two parcels—must therefore be 7'30 – 5'24 = 2'6 (edge 1–3).

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Figure 5.4

But we know that the difference is 7' and not 2'6. Lines Rev. 1–3 therefore establish that the difference 2'6 that results from the false position must be multiplied by $3^{\circ}20'$ if we are to find the true difference 7'. Since the width is already what it should be, it is the length and the "descendants" that must be multiplied by this factor. The "upper descendant" will thus be $3^{\circ}20' \cdot (30 - 18) = 40$ (line Rev. 6). Afterwards everything is quite simple; it could have been even simpler, but the road that is chosen agrees better with the pedagogical style which we know for example from TMS XVI #1, and it is probably more fruitful from a didactical point of view.

The way this problem is solved certainly differs from what we have encountered so far. But there are also common features that become more conspicuous in a bird's eye view.

The change of scale in one direction we already know as an algebraic technique. A no less conspicuous *difference*—the absence of a quadratic completion, that is, of the "Akkadian method"—points to another family characteristic: the introduction of an auxiliary figure that is first "joined" and then "torn out."

Less evident but fundamental is the "analytic" character of the methods. Since Greek antiquity, the solution of a mathematical problem is called "analytic" if it starts from the presupposition that the problem is *already* solved; that allows us to examine—"to analyze"—the characteristics of the solution in order to understand how to construct it.¹

¹The antithesis of the "analytical" method is the "synthesis, in which the solution is constructed directly, after which this solution is shown to be indeed valid." This is the proof style of Euclid's *Elements*, and since antiquity there has been the consistent complaint that this makes it more difficult than necessary to understand the work: the student sees well that each step of a proof is correct, and therefore has to accept the end result as irrefutable—but one does not understand the reasons that make the author take the single step. In this way, the author appears shrewd rather than really

A solution by equation is always analytical. In order to understand that we may look again at our modern solution of TMS XIII, the oil trade (page 70). According to the starting hypothesis, the quantity of sìLA that is bought for 1 shekel of silver is a known number, and we call it *a*. We do the same with the sales rate (which we call *v*). The total investment is hence $M \div a$, the total sales price $M \div v$, and the profit therefore $w = \frac{M}{v} - \frac{M}{a}$. Then we multiply by $v \times a$, and so forth.

That is, we treat a and v as if they were known numbers; we pretend to have a solution and we describe its characteristics. Afterwards we derive the consequences—and find in the end that a = 11, v = 7.

Even the Old Babylonian cut-and-paste solutions are analytic. Presupposing that we know a solution to the oil problem we express it as a rectangle of area 12'50, of which a part of length 40 corresponds to the oil profit. Then we examine the characteristics of this solution, and find the normalization factor by which we should multiply in order to get a difference 4 between the sides, and so on.

The solution to the present problem is also analytical. We *presuppose* that the triangle has been completed by a rectangle in such a way that the prolonged "bar" divides resulting trapezium in equal parts, after which we calculate how much the width of the rectangle must be if that shall be the case; and so on. Even though it has its justification, the distinction between "algebra" (problems that are easily translated into modern equations) and "quasi-algebra" seems less important in the perspective of the Old Babylonian texts than in ours.

BM 85200 + VAT 6599 #6

Obv. I

- 9. An excavation. So much as the length, that is the depth. 1 the dirt I have torn out. My ground and the dirt I have heaped, 1°10'. Length and width, 50'. Length, width, what?
- 10. You, 50' to 1, the conversion, raise, 50' you see. 50' to 12 raise, 10 you see.
- Make 50' confront itself, 41'40" you see; to 10 raise, 6°56'40" you see. Its IGI detach, 8'38"24" you see;
- 12. to 1°10' raise, 10'4"48" you see, 36', 24', 42' are equals.
- 13. 36' to 50' raise, 30', the length. 24' to 50' raise, 20', the width; 36' to 10 raise, 6, the depth.
- 14. The procedure.

pedagogical. Since antiquity Euclid (or his predecessors) have also been suspected to have first found their constructions and proofs by means of analysis, constructing the solution in the second instance but hiding their traces.

This is a problem of the third degree, coming from a tablet that has been broken into two parts, one of which is in London and one in Berlin (whence the composite name). It deals with a parallelepipedal "excavation," of length ℓ [NINDAN], width w [NINDAN] and depth d [κ ÙŠ]. The length is equal to the depth, but because of the use of different metrologies in the two directions that means that $d = 12\ell$.



Figure 5.5: The excavation extended 1 kùš downwards.

Further the sum of the length and the width is $[\ell + w =] 50'$, and the sum of the volume of dirt that has been "torn out," that is, dug out ² and the "ground" (the base) is $[\ell \cdot w \cdot d + \ell \cdot w =] 1^{\circ}10'$. This latter equation can be transformed into $\ell \cdot w \cdot (d + 1) = 1^{\circ}10'$ —that is, if the excavation had been dug 1 kùš deeper, its volume would have equalled $1^{\circ}10'$ [NINDAN²·kùš] (see Figure 5.5).³

The solution is based on a subtle variant of the false position (in its proper form this method would not serve, since the problem is not homogeneous—see note 7, page 50). "The position" consists in the construction of a "reference cube" with the side $\ell + w$. In horizontal measure, its side is $1 \cdot 50' = 50'$ [NINDAN],

²The text uses the same verb "to tear out" as for the subtractive operation.

³The statement also refers to "1 the dirt that I have torn out," but this information is not used. It is another example of a magnitude that is *known but not given*; knowing its numerical value allows the teacher to make a distinction between the real excavation ("1 the dirt") and the volume of the excavation extended downwards by 1 $\kappa \tilde{v}$ š ("1°10′, the dirt").

since "the conversion" of NINDAN into NINDAN asks for a multiplication by 1. In vertical measure, it is $12 \cdot 50' = 10$ kùš, since "the conversion" of NINDAN into kùš implies a multiplication by 12 (both conversions take place in line 10).

Lines 11–12 find the volume of the reference cube to be $6^{\circ}56'40''$. This volume is contained 10'4''48''' times in the extended excavation.

We should now imagine that the sides of the extended excavation are measured by the corresponding sides of the reference cube. If p is the number of times the length ℓ is measured by 50' NINDAN, q the number of times the width w is measured by 50' NINDAN, and r the number of times the depth d + 1 kùš is measured by 10 kùš (= 50' NINDAN), then

$$p \cdot 50' + q \cdot 50' = \ell + w = 50',$$

and therefore

$$p + q = 1;$$

further

$$r \cdot 10 = d + 1 = 12\ell + 1 = 12 \cdot p \cdot 50' + 1 = 10p + 1,$$

whence

$$r = p + \frac{1}{10} = p + 6';$$

and finally

$$p \cdot q \cdot r = 10'4''48'''$$

We therefore have to express 10'4''48''' as the product of three factors p, q and r that fulfil these conditions. That is what the text does in line 12, where the factors appear as the "equals" 36', 24' and 42'. Afterwards, line 13 finds ℓ , w and d.

The factorization seems to be drawn from the teacher-magician's sleeves, and that is probably how it has actually been produced, just like the various square roots and quotients. Since the solution was known beforehand, that would be easy. But it is also possible to find it by systematic reasoning, beginning with simple numbers—one must just express $10^{\circ}4^{\circ}48$ (= $2^{6} \cdot 3^{4} \cdot 7$) as the product

of three numbers P, Q and R where P + Q = 60, R = P + 6.⁴ Knowing the general character of Old Babylonian mathematics we may even claim that the text can only allow itself to draw the answer from the sleeves because it would be *possible* (albeit somewhat laborious) to find it without magic. Let us first assume that P = 1; then, since P + Q = 60, Q will be 59, which is impossible; the hypotheses P = 2 and P = 3 can be rejected for analogous reasons; P = 4 gives R = 10, which is also excluded—10^w4^v48 contains no factor 5; P = 5 is impossible in itself; P = 6 gives Q = 54 and R = 12, which must be rejected, both because the factor 7 is missing and because control shows the product not to be what is required. The next value P which does not lead to impossible values for Q or R is 12, but it must be rejected for the same reasons; P = 18 is impossible because the product is only around half of what is needed. P = 24 and P = 30 must be rejected for the same reasons as P = 6. Finally we arrive at P = 36, a value that fits. If we had counted prime factors it would have been even easier, but nothing suggests that the Babylonians knew that technique.

It must be emphasized, however, that this method only works because a simple solution *exists*. Thereby the problem differs fundamentally from those of the second degree, where a good approximation to that which "is equal" would give an almost correct solution (and the Babylonians knew well to find approximate square roots even though they did not do it in their algebra problems). The Babylonians were thus not able to solve cubic problems in general as they could solve second-degree problems—for that, one had to wait for the Italian algebraist of the sixteenth century CE.

Our text speaks of *three* "equals" which are not even equal. This usage evidently represents a generalization of an idea coming from the sides of the square and the cube. There is nothing strange in such a generalization—our own notion of the "roots" of an equation comes in the same way from early Arabic algebra, where the fundamental equations were formulated in terms of an amount of money and its square root. As this origin was forgotten the word came to be understood as a designation for the value of the unknown that satisfies the equation.

Other problems from the same tablet speak of a single "equal"; that is the case when the volume of the excavation measured by the reference parallelepiped (not always a cube) must be factorized as p^3 or as $p^2 \cdot (p+1)$. Tables indeed exist for these two functions, and in these *p* appears precisely as "the equal;" the latter table had the name "equal, 1 joined"—see page 126.

As in the second-degree algebra, the treatment of the third-degree problems is analytic—what we have just looked at is a typical representative of the category: one presupposes that a solution exists and draws the consequences from what can

⁴In order to have integers we here introduce P=60p=1'p, Q=1'q, R=1'r. Then PQR=1"pqr=10"4'48.

then be stated. In the same way, every solution by means of a false position is analytic—it begins by the hypothesis of a solution.

Apart from that, only rather peripheral characteristics connect the second and the third degree: the terminology for operations, the use of tables, the fundamental arithmetical operations.

Other problems on the same tablet (all dealing with parallelepipedal "excavations") are reduced to problems of the second or even the first degree. These are solved by the techniques we already know, and never by factorization. The Babylonians were thus aware of possessing another (and in their opinion, as we see) better technique, and they knew perfectly the difference between problems that can be solved by their algebraic techniques and those which do not yield to such attacks. But they seem not to have seen this difference as fundamental—the mathematical genre that is defined by the contents of the tablet is rather "excavation problems," just as the genre defined by BM 13901 must be understood as "square problems" even though one of the problems is reduced to a rectangle problem. Once more, the distinction between "algebra" and "quasi-algebra" seems to be secondary, less important than the classification of problems according to the object they consider.

BM 15285 #24



- 1. 1 uš the confrontation.
- 2. Inside, 16 confrontations
- 3. I have laid down. Their surface what?

The small problem that precedes is extracted from a tablet containing some 40 problems on subdivisions of a square with side 1 us = 1' NINDAN—the surviving fragments of the tablet contain 31 problems. All are accompanied by diagrams showing the actual subdivision (often necessary for understanding the sometimes very concise enunciations). Figure 5.6 shows the obverse of the principal fragment (problem #24 is found on its reverse).

The above text does not explain the procedure—none of the problems on the tablet do so. It is obvious, however, that there is no need for algebraic thinking here. It is no less evident that the technique used to calculate the coefficients in the problem BM 13901 #10 (page 46) will also serve here.

In line 3 it is seen that the verb translated "to lay down" may mean "draw," cf. note 3, page 46.



Figure 5.6: The obverse of the principal fragment of the tablet BM 15285. After C. J. Gadd, "Forms and Colours". *Revue d'Assyriologie* **19** (1922), 149–159, here p. 156.