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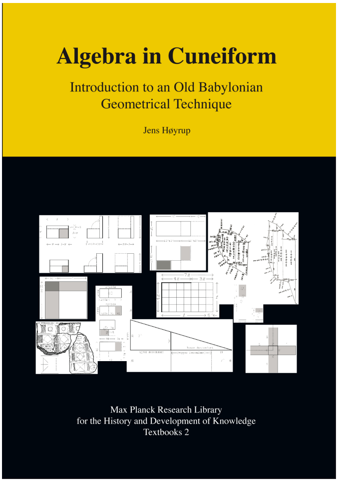
Complex Second-degree Problems



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Chapter 4
Complex Second-degree Problems

The preceding chapter set out the methods used by the Babylonians for the solution of the fundamental second-degree problems—cut-and-paste, quadratic completion, change of scale. However, as inherent in the term "fundamental," the Babylonians also worked on problems of a more complex nature. Such problems are in focus in the present chapter, which first takes up the third section of the text of which we have just examined the two introductory pedagogical sections.

TMS IX #3

19. Surface, length, and width I have heaped, 1 the surface. 3 lengths, 4 widths heaped,
20. its 17th to the width joined, 30′.
21. You, 30′ to 17 go: 8°30′ you see.
22. To 17 widths 4 widths join, 21 you see.
23. 21 as much as of widths posit. 3, of three lengths,
24. 3, as much as of lengths posit. 8°30′, what is its name?
25. 3 lengths and 21 widths heaped.
26.   8°30′ you see,
27.   3 lengths and 21 widths heaped.
28. Since 1 to the length is joined and 1 to the width is joined, make hold:
29. 1 to the heap of surface, length, and width join, 2 you see,
30. 2 the surface. Since the length and the width of 2 the surface,
31. 1°30′, the length, together with 1°20′, the width, are made hold,
32. 1 the joined of the length and 1 the joined of the width,
33. make hold, ¿1 you see.? 1 and 1, the various (things), heap, 2 you see.
34. 3…, 21…, and 8°30′ heap, 32°30′ you see;
35. so you ask.
36. … of widths, to 21, that heap:
37. … to 3, lengths, raise,
38. 1ˋ3 you see. 1ˋ3 to 2, the surface, raise:
39. 2ˋ6 you see, ¿2ˋ6 the surface?. 32°30′ the heap break, 16°15′ you ⟨see⟩.

40. {...}. 16°15′ the counterpart posit, make hold,
41. 4`24°3′45″ you see. 2`6 ¿erasure?
42. from 4`24°3′45″ tear out, 2`18°3′45″ you see.
43. What is equal? 11°45′ is equal, 11°45′ to 16°15′ join,
44. 28 you see. From the 2nd tear out, 4°30′ you see.
45. IGI 3, of the lengths, detach, 20′ you see. 20′ to 4°30′
46. {...} raise: 1°30′ you see,
47. 1°30′ the length of 2 the surface. What to 21, the widths, may I posit
48. which 28 gives me? 1°20′ posit, 1°20′ the width
49. of 2 the surface. Turn back. 1 from 1°30′ tear out,
50. 30′ you see. 1 from 1°20′ tear out,
51. 20′ you see.

Lines 19 and 20 present a system of two equations about a rectangle, one of the first and one of the second degree. The former is of the same type as the one explained in TMS XVI #1 (see page 27). The second coincides with the one that was examined in section #2 of the present text (see page 54). In symbolic translation, the equation system can be written

$$\tfrac{1}{17}(3\ell + 4w) + w = 30' \quad , \quad \sqsubset\!\sqsupset(\ell, w) + \ell + w = 1.$$

In agreement with what we have seen elsewhere, the text multiplies the first-degree equation by 17 (using the Akkadian verb "to go," see page 19), thus obtaining integer coefficients (*as much as*):

$$3\ell + (4 + 17)w = 3\ell + 21w = 17 \cdot 30' = 8°30'.$$

This is done in the lines 21–25, while the lines 26 and 27 summarize the result.

Lines 28–30 repeat the trick used in section #2 of the text (see Figure 3.10): the length and the width are prolonged by 1, and the square that is produced when that which the two "joined"[1] "hold" is "joined" to the "heap" $\sqsubset\!\sqsupset(\ell, w) + \ell + w$; out of this comes a "surface 2," the meaning of which is again explained in lines 30–33.

The lines 34–37 are very damaged, too damaged to be safely reconstructed as far as their words are concerned. However, the numbers suffice to see how the calculations proceed. Let us introduce the magnitudes $\lambda = \ell + 1$ and $\phi = w + 1$. The text refers to them as the length and width "of the surface 2"—in other words, $\sqsubset\!\sqsupset(\lambda, \phi) = 2$. Further,

[1] As the "to-be-joined" of page 37, this noun (*wuṣubbûm*) is derived from the verb "to join."

$$\begin{aligned} 3\lambda + 21\phi &= 3 \cdot (\ell + 1) + 21 \cdot (w + 1) \\ &= 3 + 21 + 3\ell + 21w \\ &= 3 + 21 + 8°30' \\ &= 32°30'. \end{aligned}$$

In order to facilitate the understanding of what now follows we may further introduce the variables

$$L = 3\lambda \quad , \quad W = 21\phi$$

(but we must remember that the text has no particular names for these—in contrast to λ and ϕ which do have names; we now speak *about*, not *with* the Babylonian author). Lines 36–39 find that

$$\sqsubset\!\sqsupset(L, W) = (21 \cdot 3) \cdot 2 = 1`3°2' = 2`6°;$$

summing up we thus have

$$L + W = 32°30' \quad , \quad \sqsubset\!\sqsupset(L, W) = 2`6°.$$

We have now come to line 39, and arrived at a problem type which we had not seen so far: A rectangle for which we know the area and the *sum* of the two sides.

Once again, a cut-and-paste method is appealed to (see Figure 4.1). As before, the known segment is "broken" together with the rectangle which goes with it. In the present situation, this segment is the sum of L and W. This rectangle is composed from $\sqsubset\!\sqsupset(L, W)$, traced in full, and a square $\square(L)$ to its right, drawn with a dotted line. Next, we let the two "moieties" of this segment "hold" a square (lines 39–40). As we see, that part of the original rectangle $\sqsubset\!\sqsupset(L, W)$ which falls outside the new square can just be fitted into it so as to form a gnomon together with that part which stays in place. In its original position, this piece appears in light shading, whereas it is darkly shaded in its new position.

One part of the new square $\square(16°15')$ is constituted by the gnomon, whose area results from recombination of the original rectangle $\sqsubset\!\sqsupset(L, W)$; this area is hence 2`6. We also know the area of the outer square, $16°15' \times 16°15 = 4`24°3'45''$ (lines 40 and 41). When the gnomon is "torn out" (lines 41 and 42), $2`18°3'45''$ remains for the square contained by the gnomon. Its side (that which "is equal") is $11°45'$, which must now be "joined" to one of the pieces $16°15'$ (which gives us W) and "torn out" from the other, its "counterpart" (which gives us L). This time, however, it is not *the same* piece that is "joined" and "torn out";

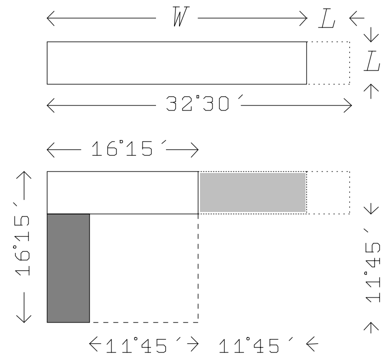

Figure 4.1: The cut-and-paste method of TMS IX #3.

there is hence no reason to "tear out" before "joining," as in YBC 6967 (page 46), and the normal priority of addition can prevail. Lines 43–44 find $W = 28$ and $L = 4°30'$. Finally, the text determines first λ and ϕ and then ℓ and w—we remember that $L = 3\lambda, \lambda = \ell + 1, W = 21\phi, \phi = w + 1$. Since 28 has no IGI, line 48 explains that $21 \cdot 1°20' = 28$.

AO 8862 #2

I

30. Length, width. Length and width
31. I have made hold: A surface I have built.
32. I turned around (it). The half of the length
33. and the third of the width
34. to the inside of my surface
35. I have joined: 15.
36. I turned back. Length and width
37. I have heaped: 7.

II

1. Length and width what?
2. You, by your proceeding,
3. 2 (as) inscription of the half
4. and 3 (as) inscription
5. of the third you inscribe:

6. IGI 2, 30′, you detach:
7. 30′ steps of 7, 3°30′; to 7,
8. the things heaped, length and width,
9. I bring:
10. 3°30′ from 15, my things heaped,
11. cut off:
12. 11°30′ the remainder.
13. Do not go beyond. 2 and 3 make hold:
14. 3 steps of 2, 6.
15. IGI 6, 10′ it gives you.
16. 10′ from 7, your things heaped,
17. length and width, I tear out:
18. 6°50′ the remainder.
19. Its moiety, that of 6°50′, I break:
20. 3°25′ it gives you.
21. 3°25′ until twice
22. you inscribe; 3°25′ steps of 3°25′,
23. 11°40′25″; from the inside
24. 11°30′ I tear out:
25. 10′25″ the remainder. ⟨By 10′25″, 25′ is equal⟩.
26. To the first 3°25′
27. 25′ you join: 3°50′,
28. and (that) which from the things heaped of
29. length and width I have torn out
30. to 3°50′ you join:
31. 4 the length. From the second 3°25′
32. 25′ I tear out: 3 the width.

32a. 7 the things heaped.

32b. 4, the length; 3, the width; 12, the surface.

The first two words of the first line (I.30) tell us that we are dealing with a figure that is fully characterized by its length and its width, that is, with a rectangle (cf. page 28)—or rather with a rectangular field: references to surveyors' practice can be found in the text (for instance, *I turned around it* in line I.32 probably means that the surveyor, after having laid out a field, has walked around it; in I.36 he *turned back*).

Before studying the procedure, we may concentrate on certain aspects of the formulation of the text. In line I.31 we see that the operation "to make hold" does not immediately produce a numerical result—since the measures of the sides are still unknown, that would indeed be difficult. The text only says that a "surface" has been "built"; we are probably meant to understand that it has been laid out in the terrain. Later, when two known segments are to "hold" (lines II. 13–14, and perhaps II.21–22), the numerical determination of the area appears as a distinct operation, described with the words of the table of multiplication. Finally, we observe that the text defines the outcome of a "heaping" addition as a plural, translated "the things heaped," and that the normal alternating pattern of grammatical person is not respected.

The text, almost certainly from Larsa, seems to be from c. 1750 BCE and thus to belong to the early phase of the adoption of algebra by the southern scribe school (see page 109). These particularities may therefore give us information about the ideas on which it was based—such ideas were to become less visible once the language and format became standardized.

The topic of the problem is thus a rectangle. Lines I.36–37 tell us that the "heap" of its length and width is 7, while the lines I. 32–35 state that "joining" half of the length and one third of the width to the "surface" produces 15:[2]

$$\sqsubset\!\sqsupset(\ell, w) + \tfrac{1}{2}\ell + \tfrac{1}{3}w = 15 \quad , \quad \ell + w = 7.$$

The upper part of Figure 4.2 illustrates this situation, with 2 and 3 "inscribed as inscription" of $\frac{1}{2}$ respectively $\frac{1}{3}$ of the "projections"[3] 1 of the length and the width (lines II.2–5); the heavily drawn configuration thus has an area equal to 15.

The solution *could* have followed the pattern of TMS IX #3 (page 57). By introducing an "extended length" $\lambda = \ell + \frac{1}{3}$ and an "extended width" $\phi = w + \frac{1}{2}$, and adding (according to the "Akkadian method") the rectangle $\sqsubset\!\sqsupset(\frac{1}{2}, \frac{1}{3})$ which is lacking in the corner where 2 and 3 are "inscribed," we would have reduced the problem to

$$\sqsubset\!\sqsupset(\lambda, \phi) = 15 + \sqsubset\!\sqsupset\left(\tfrac{1}{2}, \tfrac{1}{3}\right) = 15°10', \lambda + \phi = 7 + \tfrac{1}{2} + \tfrac{1}{3} = 7°50'.$$

[2]We should observe that the half that appears here is treated as any other fraction, on an equal footing with the subsequent third. It is not a "moiety," and the text finds it through multiplication by 30′, not by "breaking."

Let us also take note that the half of the length and the third of the width are "joined" to the "surface," not "heaped" together with it. A few other early texts share this characteristic. It seems that the surveyors thought in terms of "broad lines," strips possessing a tacitly understood breadth of 1 length unit; this practice is known from many pre-Modern surveying traditions, and agrees well with the Babylonian understanding of areas as "thick," provided with an implicit height of 1 KÙŠ (as inherent in the metrology of volumes, which coincides with that for areas—see page 17). The "projection" and "base" of BM 13901 and TMS IX #1 are likely to be secondary innovations due to the school—different schools, indeed, and therefore different words. They allowed segments to be thought of as truly one-dimensional while still permitting their transformation into rectangles with width 1.

[3]The absence of this notion from the text should not prevent it from using it as a technical term of general validity.

However, the present text does not proceed like that—Old Babylonian algebra was a flexible instrument, not a collection of recipes or algorithms to be followed to the letter. The text finds the half of 7 (that is, of the sum of the length and the width) and "brings" the outcome 3°30′ to "the things heaped, length and width." "To bring" is no new arithmetical operation—the calculation comes afterwards. The text must be understood literally, the rectangle $\sqsubset\!\sqsupset(\ell + w, \frac{1}{2})$ (represented by the number 3°30′) is brought *physically* to the place where length and width (provided with widths $\frac{1}{2}$ and $\frac{1}{3}$) are to be found. In this way it becomes possible to "cut off" the rectangle $\sqsubset\!\sqsupset(\ell + w, \frac{1}{2})$—as long as it was elsewhere that would make no sense. In bottom of Figure (4.2), the area that is eliminated is drawn shaded and black: the rest, in white, will be equal to 11°30′.

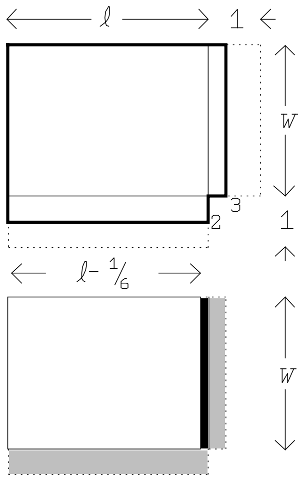

Figure 4.2: The reduction of AO 8862 #2.

In this operation, it is obvious that the (shaded) half of the length that had been "joined" according to the statement has been eliminated. However, more than the (equally shaded) third of the width has disappeared. How much more precisely?

It would be easy to subtract 20′(= $\frac{1}{3}$) from 30′(= $\frac{1}{2}$), but that may not have been deemed sufficiently informative.[4] In any case, the text introduces a detour

[4] Alternatively, the trick used by the text could be a leftover from the ways of surveyors not too familiar with the place-value system; or (a third possibility) the floating-point character of this system might make it preferable to avoid it in contexts where normal procedures for keeping track of orders of magnitude (whatever these normal procedures were) were not at hand.

by the phrase "Do not go beyond!" (the same verb as in the "subtraction by comparison"). A rectangle ⊏⊐(3,2) is constructed (perhaps one should imagine it in the corner where 2 and 3 are "inscribed" in Figure 4.2; in any case Figure 4.3 shows the situation). Without further argument it is seen that the half (three small squares) exceeds the third (two small squares) by one of six small squares, that is, by a sixth—another case of reasoning by "false position." Exceptionally, IGI 6 is not "detached" but "given" (namely by the table of reciprocals).

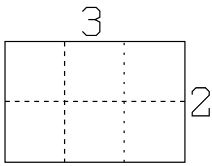

Figure 4.3

We thus know that, in addition to the third of the width, we have eliminated a piece ⊏⊐(w,10′) (drawn in black); if $\lambda = \ell - 10'$, we therefore have

$$\lambda + w = 7 - 10' = 6°50' \quad , \quad ⊏⊐(\lambda, w) = 11°30'.$$

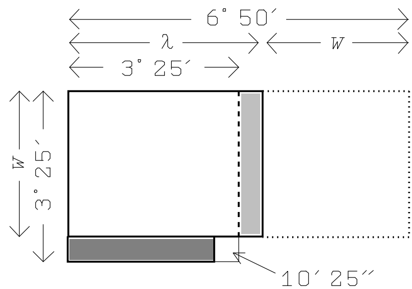

Figure 4.4

Once more we therefore have a rectangle of which we know the area and the sum of length and width. The procedure is the same as in the final part of TMS IX #3—see Figure 4.4; the area that is to be displaced is shown again in light shading in the position from where it is to be taken and in heavy shading where it has to be placed. The only difference is terminological: in TMS IX #3, the two "moieties" are "made hold," here they are "inscribed"—but since a multiplication of a number by a number follows immediately, the usual construction of a rectangle (here a square) must be intended (lines II.13–14).[5]

In the end, the final addition of the side of the square precedes the subtraction, as in TMS IX #3. Once more, indeed, it is not the same piece that is involved in the two operations; there is therefore no need to make it available before it is added.

VAT 7532

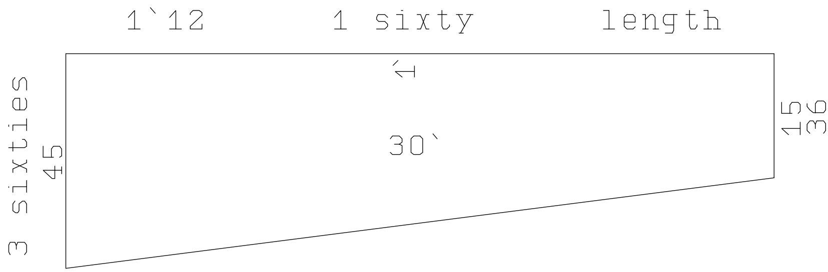

Figure 4.5: The diagram of VAT 7532. The "upper width" is to the left.

Obv.

1. A trapezium. I have cut off a reed. I have taken the reed, by its integrity
2. 1 sixty (along) the length I have gone. The 6th part
3. broke off for me: 1`12 to the length I have made follow.
4. I turned back. The 3rd part and $\frac{1}{3}$ KÙŠ broke off for me:
5. 3 sixty (along) the upper width I have gone.
6. With that which broke off for me I enlarged it:
7. 36 (along) the width I went. 1 BÙR the surface. The head (initial magnitude) of the reed what?
8. You, by your proceeding, (for) the reed which you do not know,
9. 1 may you posit. Its 6th part make break off, 50′ you leave.
10. IGI 50′ detach, 1°12′ to 1 sixty raise:

[5] It is not quite to be excluded that the text does not directly describe the construction but refers to the inscription twice of 3°25′ on a tablet for rough work, followed by the numerical product—cf. above, note 11, page 21; in that case, the construction itself will have been left implicit, as is the numerical calculation in other texts. Even the "inscription" of 2, followed by its IGI (II.3 and 6) might refer to this type of tablet. Then, however, one would expect that the "detachment" of the IGI should follow the inscription immediately; moreover, the inscription of 3 in line II.4 is not followed at all by "detachment" of its IGI, which after all speaks against this reading of the lines II.3–6 and II.21–22.

11. 1ˋ12 to ⟨1ˋ12⟩ join: 2ˋ24 the false length it gives you.
12. (For) the reed which you do not know, 1 may you posit. Its 3rd part make break off,
13. 40′ to 3 sixty of the upper width raise:
14. 2ˋ it gives you. 2ˋ and 36 the lower width heap,
15. 2ˋ36 to 2ˋ24 the false length raise, 6ˋˋ14ˋ24 the false surface.
16. The surface to 2 repeat, 1ˋˋ to 6ˋˋ14ˋ24 raise
17. 6ˋˋˋˋ14ˋˋˋ24ˋˋ it gives you. And $\frac{1}{3}$ KÙŠ which broke off
18. to 3 sixty raise: 5 to 2ˋ24, the false length,
19. raise: 12ˋ. $\frac{1}{2}$ of 12ˋ break, 6ˋ make encounter,

Rev.

1. 36ˋˋ to 6ˋˋˋˋ14ˋˋˋ24ˋˋ join, 6ˋˋˋˋ15ˋˋˋ it gives you.
2. By 6ˋˋˋˋ15ˋˋˋ, 2ˋˋ30′ is equal. 6ˋ which you have left
3. to 2ˋˋ30ˋ join, 2ˋˋ36ˋ it gives you. IGI 6ˋˋ14ˋ24,
4. the false surface, I do not know. What to 6ˋˋ14ˋ24
5. may I posit which 2ˋˋ36 gives me? 25′ posit.
6. Since the 6th part broke off before,
7. 6 inscribe: 1 make go away, 5 you leave.
8. ⟨IGI 5 detach, 12′ to 25 raise, 5′ it gives you⟩. 5′ to 25′ join: $\frac{1}{2}$ NINDAN, the head of the reed it gives you.

This problem also deals with a field—yet with a field which the surveyor would only encounter in dream (or rather, in a nightmare). "Real life" enters through the reference to the unit BÙR, a unit belonging to practical agricultural administration, and through the reference to measuring by means of a reed cut for this purpose; its length ($\frac{1}{2}$ NINDAN) corresponds indeed to a measuring unit often used in practical life and called precisely a "reed" (GI in Sumerian). One may also imagine that such reeds would easily break. Finally, the use of the numeral "sixty" shows us one of the ways to express numbers unambiguously.

Everything else, however—that is, that the area of the field is known before it is measured, and also the ways to indicate the measures of the pieces that break off from the reed—shows which ruses the Old Babylonian school masters had to make use of in order to produce second-degree problems having some taste of practical life.

For once, Figure 4.5 reproduces a diagram that is traced on the tablet itself. In general, as also here, diagrams are only drawn on the tablets when they serve to clarify the statement; they are never used to explain the procedure. On the other hand, Figure 4.5 shows once more that the solution is known in advance: the numbers 1ˋ, 45 and 15 are indeed the measures of the sides expressed in NINDAN.

We thus undertake to measure the trapezium by means of a reed of unknown length R. We manage to measure 1` reed lengths along the length of the trapezium before the reed loses a sixth of its length and is reduced to $r = \frac{5}{6}R$. What remains of the length turns out to be 1`12r (lines Obv. 2–3).

Then the reed breaks for the second time. According to lines Obv. 4 and 5, the measure of the "upper width" (to the left)[6] is 3`z, where $z = \frac{2}{3}r - \frac{1}{3}$ KÙŠ is the length of the reed after this second reduction.

The piece that broke off last is put back into place, and the "(lower) width" (evidently to the right) is paced out (line Obv. 7) as 36 r. Finally we learn that the area of the field is 1 BÙR = 30` SAR (1 SAR = 1 NINDAN^2), see page 17). We are asked to find the original length of the reed—its "head" in the sense of "beginning."

Lines Obv. 9–11 determine the length in units r by means of a false position: if R had been equal to 1, then r would have been 50′; conversely, R must correspond to r multiplied by IGI 50′ = 1°12′. 1` steps of $R$ thus correspond to 1`12 · r, and the complete length will be

$$1`12 \cdot r + 1`12 \cdot r = 2`24 \cdot r.$$

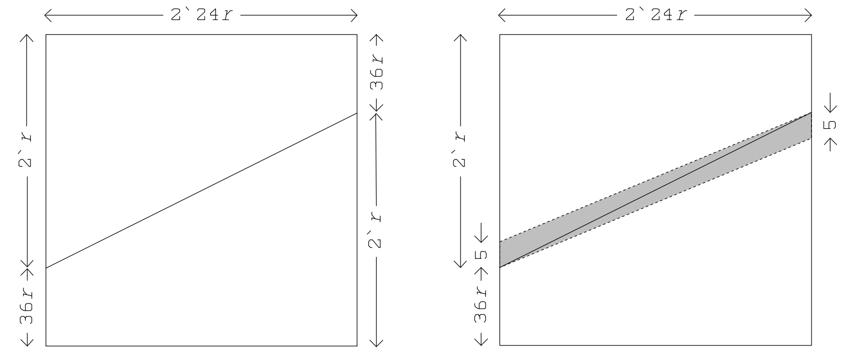

Figure 4.6: The doubled trapezium of VAT 7532.

[6]The position of the "upper" width to the left is a consequence of the new orientation of the cuneiform script (a counterclockwise rotation of 90°) mentioned in the box "Cuneiform writing." On tablets, this rotation took place well before the Old Babylonian epoch, as a consequence of which one then wrote from left to right. But Old Babylonian scribes knew perfectly well that the *true* direction was vertically downwards—solemn inscriptions on stone (for example Hammurabi's law) were still written in that way. For reading, scribes may well have turned their tablets 90° clockwise.

The text speaks of 2`24 as the "false length," that is, the length expressed in units *r*.

Another false position is applied in line Obv. 12. The text posits 1 for the length *r* of the reed once shortened, and deducts that what remains after the loss of $\frac{1}{3}$ must be equal to 40′. Leaving aside the extra loss of $\frac{1}{3}$ KÙŠ, the false upper width (the upper width measured in units *r*) is thus 40′ times 3 sixties, that is, $40' \cdot 3` = 2`$. In other words, the upper width measures 2`*r*—still leaving aside the missing piece of $\frac{1}{3}$ KÙŠ.

Since line Obv. 7 indicates that the false (lower) width is 36, we thus know—with the same reserve concerning the missing $\frac{1}{3}$ KÙŠ—the three sides that will allow us to determine the area of the trapezium in units □(*r*).

Yet the text does not calculate this area: *The surface to 2 repeat.* Instead it doubles the trapezium so as to form a rectangle (see the left part of Figure 4.6), and the lines Obv. 14–16 calculate the area of this rectangle (the "false surface"), finding 6``14`24 (in the implicit unit □(*r*)).

If the reed had not lost an ulterior piece of $\frac{1}{3}$ KÙŠ, we might now have found the solution by means of a final false position similar to that of BM 13901 #10 (see page 46): according to line Obv. 7, the area of the field is 1 BÙR, the doubled area hence 2 BÙR = 1`` NINDAN$^2$ (Obv. 16: *The surface to 2 repeat,* 1``). However, things are more complicated here. For each of the 3` steps made by the twice shortened reed a piece of $\frac{1}{3}$ KÙŠ is missing from our calculation, in total thus $3` \cdot \frac{1}{3}$ KÙŠ = 1` KÙŠ = 5 NINDAN (1 KÙŠ = $\frac{1}{12}$ NINDAN): *And $\frac{1}{3}$ KÙŠ which broke off to 3 sixty raise: 5* (Obv. 17–18). Therefore the area of the real field does not correspond to what we see to the left in Figure 4.6 but to that which remains after elimination of the shaded strip to the right. The area of this strip is 5·2`24*r* = 12`*r*: *5 to 2`24, the false length, raise: 12`*. The relation between the "false surface" and that of the doubled real trapezium can now be expressed by the equation

$$6``14`24\square(r) - 12`r = 1``.$$

This non-normalized equation is solved in the usual way. First it is multiplied by 6``14`24: 1`` *to* 6``14`24 *raise* 6````14```24`` *it gives you* (Obv. 16–17). That leads to the normalized equation

$$\square(6``14`24r) - 12` \cdot (6``14`24r) = 6````14```24``$$

or, with $s = 6``14`24r$ *r* as unknown,

$$\square(s) - 12s = 6````14```24``.$$

From here onward, the procedure coincides with that of BM 13901 #2 (page 43), with a small variation in the end. The calculations can be followed in Figure 4.7.

The area $6^{\backprime\backprime\backprime\backprime}14^{\backprime\backprime\backprime}24^{\backprime\backprime}$ corresponds to the rectangle of (height) s and breadth $s - 12^{\backprime}$. Half of the excess of the height over the breadth is "broken" and repositioned as seen in the diagram: lightly shaded in the original positions, heavily shaded where it is moved to. The construction of the completing square is described with one of the synonyms of "making hold," namely "to make encounter" (Obv. 19).

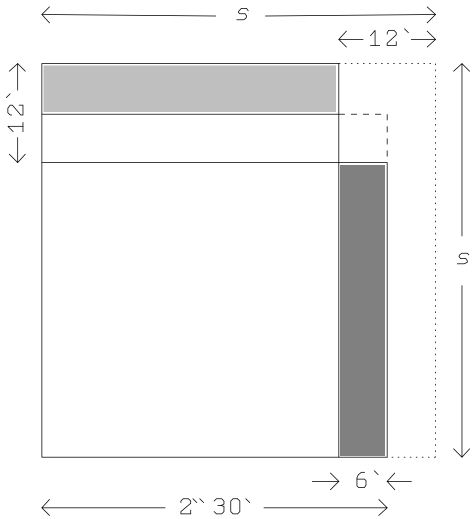

Figure 4.7

After the usual operations we find that $s = 6^{\backprime\backprime}14^{\backprime}24$ $r = 2^{\backprime\backprime}36^{\backprime}$, and in line Rev. 5 that $r = 25'$. We observe, however, that the "moiety" that was moved around is not put back into its original position, which would have reconstituted s in the vertical direction. Instead, the other "moiety," originally left in place, is also moved, which allows a horizontal reconstitution $s = 6^{\backprime\backprime}14^{\backprime}24$ $r = 2^{\backprime\backprime}36$: *6`which you have left to 2``30` join, 2``36` it gives you.*[7]

In the lines Rev. 6–8, the calculator introduces a third false position: if R had been equal to 6, then r would be 5. The difference of 1 between R and r is $\frac{1}{5}$ of r or $12'$ times r. Now the true value of r is $25'$; in order to obtain R we must hence "join" $12' \cdot 25' = 5'$ to it. Therefore $R = 25' + 5' = 30' = \frac{1}{2}$ NINDAN.

[7]This distinction between two halves of which one is "left" is worth noticing as another proof of the geometric interpretation—it makes absolutely no sense unless understood spatially.

One might believe this problem type to be one of the absolute favorites of the Old Babylonian teachers of sophisticated mathematics. We know four variants of it differing in the choice of numerical parameters. However, they all belong on only two tablets sharing a number of terminological particularities—for instance, the use of the logogram $\frac{1}{2}$ for the "moiety," and the habit that results are "given," not (for example) "seen" or "coming up." Both tablets are certainly products of the same locality and local tradition (according to the orthography based in Uruk), and probably come from the same school or even the same hand. A simpler variant with a rectangular field, however, is found in an earlier text of northern origin, and also in a text belonging together with the trapezium variants; if not *the* favorite, the broken reed was probably *a* favorite.

TMS XIII

As TMS VII #2, this problem is rather difficult. It offers an astonishing example of application of the geometrical technique to a non-geometrical question.

1. 2 GUR 2 PI 5 BÁN of oil I have bought. From the buying of 1 shekel of silver,
2. 4 SILÀ, each (shekel), of oil I have cut away.
3. $\frac{2}{3}$ mina of silver as profit I have seen. Corresponding to what
4. have I bought and corresponding to what have I sold?
5. You, 4 SILÀ of oil posit and 40, (of the order of the) mina, the profit posit.
6. IGI 40 detach, 1′30″ you see, 1′30″ to 4 raise, 6′ you see.
7. 6′ to 12‵50, the oil, raise, 1‵17 you see.
8. $\frac{1}{2}$ of 4 break, 2 you see, 2 make hold, 4 you see.
9. 4 to 1‵17 join, 1‵21 you see. What is equal? 9 is equal.
10. 9 the counterpart posit. $\frac{1}{2}$ of 4 which you have cut away break, 2 you see.
11. 2 to the 1st 9 join, 11 you see; from the 2nd tear out,
12. 7 you see. 11 SILÀ each (shekel) you have bought, 7 SILÀ you have sold.
13. Silver corresponding to what? What to 11 ¿SILÀ? may I posit
14. which 12‵50 of oil gives me? 1‵10 posit, 1 mina 10 shekel of silver.
15. By 7 SILÀ each (shekel) which you sell of oil,
16. that of 40 of silver corresponding to what? 40 to 7 raise,
17. 4‵40 you see, 4‵40 of oil.

This is another problem which, at superficial reading, seems to reflect a situation of real practical (here, commercial) life. At closer inspection, however, it turns out to be just as artificial as the preceding broken-reed question: a merchant has bought $M = 2$ GUR 2 PI 5 BÁN (= 12‵50 SÌLA) of fine oil (probably sesame oil). We are not told how much he paid, but the text informs us that from the quantity of oil which he has bought for one shekel (a) he has cut away 4 sìla, selling what was left ($v = a - 4$) for 1 shekel; a and v are thus the reciprocals of the two prices—we may speak of them as "rates" of purchase and sale. Moreover, the total profit w amounts to $\frac{2}{3}$ *mina* = 40 shekel of silver. For us, familiar with algebraic letter symbolism, it is easy to see that the total purchase price (the investment) must be $M \div a$, the total sales price $M \div v$, and the profit in consequence $w = (M \div v) - (M \div a)$. Multiplying by $a \cdot v$ we thus get the equation

$$M \cdot (a - v) = w \cdot av,$$

and since $v = a - 4$, the system

$$a - v = 4 \quad , \quad a \cdot v = (4M) \div w.$$

This system—of the same type as the one proposed in YBC 6967, the *igûm-igibûm* problem (page 46)—is indeed the one that is solved from line 8 onward. Yet it has certainly not been reached in the way just described: on one hand because the Babylonians did not have our letter symbolism, on the other because they would then have found the magnitude $(4M) \div w$ and not, as they actually do, $(4 \div w) \cdot M$.

The cue to their method turns up towards the end of the text. Here the text first finds the total investment and next the profit *in oil* (4`40 sìla). These calculations do not constitute a proof since these magnitudes are not among the data of the problem. Nor are they asked for, however. They must be of interest because they have played a role in the finding of the solution.

Figure 4.8 shows a possible and in its principles plausible interpretation. The total quantity of oil is represented by a rectangle, whose height corresponds to the total sales price in shekel, and whose breadth is the "sales rate" v (sìla per shekel). The total sales price can be divided into profit (40 shekel) and investment (purchase price), and the quantity of oil similarly into the oil profit and the quantity whose sale returns the investment.

The ratio between the latter two quantities must coincide with that into which the quantity bought for one shekel was divided—that is, the ratio between 4 sìla and that which is sold for 1 shekel (thus v).

Modifying the vertical scale by a factor which reduces 40 to 4, that is, by a factor $4 \div w = 4 \div 40 = 6'$, the investment will be reduced to v, and the area to $(4 \div w) \cdot M = 1`17$. In this way we obtain the rectangle to the right, for which we know the area ($a \cdot v = 1`17$) and the difference between the sides ($a - v = 4$), exactly as we should. Moreover, we follow the text in the order of operations, and the oil profit as well as the investment play a role.

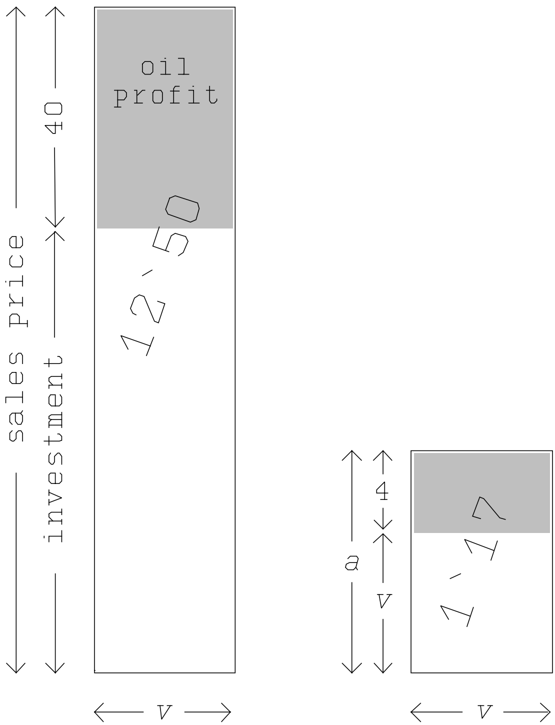

Figure 4.8: Geometric representation of TMS XIII.

On the whole, the final part of the procedure follows the model of YBC 6967 (and of other problems of the same type). The only difference occurs in line 10: instead of using the "moiety" of $a-v$ which we have "made hold" in line 8, $a-v$ is "broken" a second time. That allows us to "join" first (that which is joined is already at disposal) and to "tear out" afterwards.

In YBC 6967, the *igûm-igibûm* problem (page 46), the geometric quantities served to represent magnitudes of a different nature, namely abstract numbers. Here, the representation is more subtle: one segment represents a quantity of silver, the other the quantity of oil corresponding to a shekel of silver.

BM 13901 #12

Obv. II

27. The surfaces of my two confrontations I have heaped: $21'40''$.
28. My confrontations I have made hold: $10'$.
29. The moiety of $21'40''$ you break: $10'50''$ and $10'50''$ you make hold,
30. $1'57''21{+}25'''40''''$[8] is it. $10'$ and $10'$ you make hold, $1'40''$
31. inside $1'57''21\{+25\}'''40''''$ you tear out: by $17'21\{+25\}'''40''''$, $4'10''$ is equal.
32. $4'10''$ to one $10'50''$ you join: by $15'$, $30'$ is equal.
33. $30'$ the first confrontation.
34. $4'10''$ inside the second $10'50''$ you tear out: by $6'40''$, $20'$ is equal.
35. $20'$ the second confrontation.

With this problem we leave the domain of fake practical life and return to the geometry of measured geometrical magnitudes. However, the problem we are going to approach may confront us with a possibly even more striking case of representation.

This problem comes from the collection of problems about squares which we have already drawn upon a number of times. The actual problem deals with two squares; the sum of their areas is given, and so is that of the rectangle "held" by the two "confrontations" c_1 and c_2 (see Figure 4.9):

$$\square(c_1) + \square(c_2) = 21'40'' \quad , \quad \sqsubset\!\sqsupset(c_1, c_2) = 10'.$$

The problem could have been solved by means of the diagram shown in Figure 4.10, apparently already used to solve problem #8 of the same tablet, which can be expressed symbolically as follows:

$$\square(c_1) + \square(c_2) = 21'40'' \quad , \quad c_1 + c_2 = 50'.$$

However, the author chooses a different method, showing thus the flexibility of the algebraic technique. He takes the two *areas* $\square(c_1)$ and $\square(c_2)$ as *sides* of

[8]By error, line 30 of the text has $1'57''46'''40''''$ instead of $1'57''21'''40''''$; a partial product 25 has been inserted an extra time, which shows that the computation was made on a separate device where partial products would disappear from view once they had been inserted. This excludes writing on a clay surface and suggests instead some kind of reckoning board.

The error is carried over in the following steps, but when the square root is taken it disappears. The root was thus known in advance.

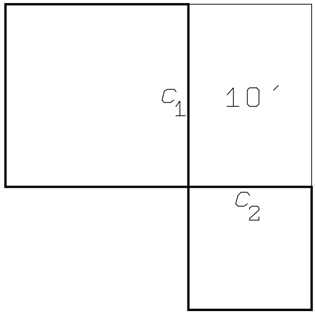

Figure 4.9: The two squares and the rectangle of BM 13901 #12.

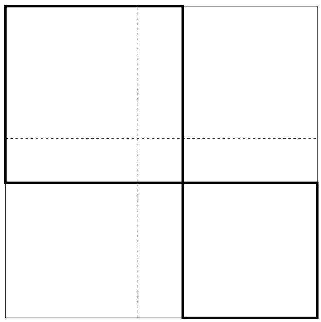

Figure 4.10: The diagram that corresponds to BM 13901 #8.

a rectangle, whose area can be found by making $10'$ and $10'$ "hold" (see Figure 4.10):

$$\square(c_1) + \square(c_2) = 21'40'' \quad , \quad \sqsubset\!\sqsupset(\square(c_1), \square(c_2)) = 10' \times 10' = 1'40''.$$

In spite of the geometric character of the operations the Babylonians were thus quite aware that the area of a rectangle whose sides are the squares $\square(c_1)$ and $\square(c_2)$ coincides with that of a square whose side is the rectangle $\sqsubset\!\sqsupset(c_1, c_2)$— which corresponds to our arithmetical rule $p^2 \cdot q^2 = (pq)^2$.

We now have a rectangle for which we know the area and the sum of the two sides, as in the problems TMS IX #3 (page 57) and AO 8862 #2 (page 60).

The solution follows the same pattern, but with one inevitable difference: this procedure can only give us □(c_1) and □(c_2); in order to know c_1 and c_2 we must find out what "is equal by" them. The calculations can be followed on Figure 4.11.

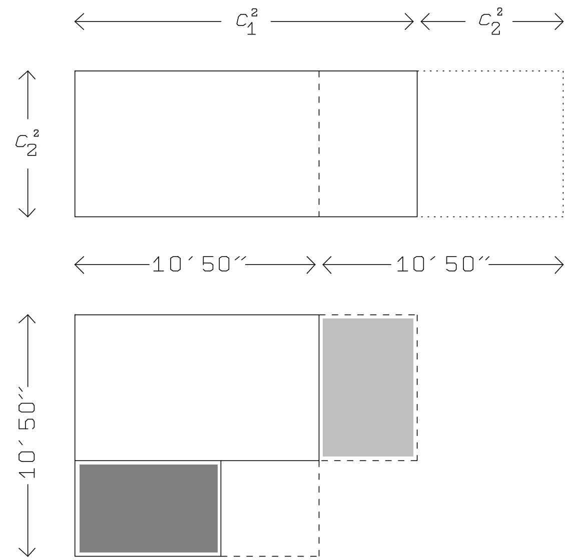

Figure 4.11: The procedure used to solve the rectangle problem.

What is to be taken note of in this problem is hence that it represents *areas* by line segments and the *square of an area* by an area. Together with the other instances of representation we have encountered, the present example will allow us to characterize the Old Babylonian technique as a *genuine algebra* on page 99.

BM 13901 #23

Rev. II

11. About a surface, the four widths and the surface I have heaped, 41′40″.
12. 4, the four widths, you inscribe. IGI 4 is 15′.
13. 15′ to 41′40″ you raise: 10′25″ you inscribe.
14. 1, the projection, you join: by 1°10′25″, 1°5′ is equal.
15. 1, the projection, which you have joined, you tear out: 5′ to two
16. you repeat: 10′, NINDAN, confronts itself.

Whereas the previous problem illustrates the “modern” aspect of Old Babylonian mathematics, the present one seems to illustrate its archaic side—even though they come from the same tablet.

This is no real contradiction. The present problem #23 is *intentionally* archaic. In other words, it is *archaizing* and not truly archaic, which explains its appearance together with the “modern” problems of the same collection. The author is not modern and archaic at the same time, he shows his virtuosity by playing with archaisms. In several ways, the formulations that are used here seem to imitate the parlance of Akkadian surveyors. The text speaks of the *width* of a square, not of a “confrontation”; further, this word appears in syllabic writing, which is quite exceptional (cf. note 4, page 16). The introductory phrase “About a surface”[9] seems to be an abbreviated version of the characteristic formula introducing a mathematical riddle: “if somebody asks you thus about a surface ...” (cf. pages 34, 110, 111 and 127). The expression “the four widths”[10] reflects an interest in what is *really there* and for what is *striking*, an interest that characterizes riddles in general but also the mathematical riddles that circulated among the mathematical practitioners of the pre-Modern world (see page 106). Even the method that is used is typical of riddles: the use of an astonishing artifice that does not invite generalization.

The problem can thus be expressed in the following way:

$$4c + \square(c) = 41'40''.$$

Figure 4.12 makes clear the procedure: $4c$ is represented by 4 rectangles $\sqsubset\!\sqsupset(1, c)$; the total $41'40''$ thus corresponds to the cross-shaped configuration where a “projection” protrudes in each of the four principal directions.

Lines 12–13 prescribe to cut out $\frac{1}{4}$ of the cross (demarcated by a dotted line) and the “joining” of a quadratic complement $\square(1)$ to the gnomon that results. There is no need to “make hold,” the sides of the complement are already there in the right position. But it is worthwhile to notice that it is the “projection” itself that is “joined”: it is hence no mere number but a quadratic configuration identified by its side.

[9] In the original, the word is “surface” marked by a phonetic complement indicating the accusative. An accusative in this position is without parallel, and seems to allow no interpretation but the one given here.

[10] For once, the determinate article corresponds to the Akkadian, namely to an expression which is only used to speak about an inseparable plurality (such as “the four quarters of the world” or “the seven mortal sins”).

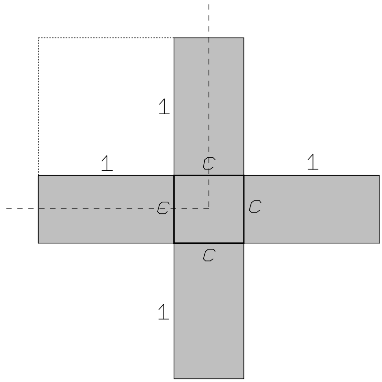

Figure 4.12: The procedure of BM 13901 #23.

The completion of the gnomon gives a square with area 1°10′25″ and thus side 1°5′. "Tearing out" the "projection"—now as a one dimensional entity—we find 5′. Doubling this result, we get the side, which turns out to be 10′. Here again, the text avoids the usual term and does not speak of a "confrontation" as do the "modern" problems of the collection; instead it says that 10′ NINDAN "confronts itself."

This method is so different from anything else in the total corpus that Neugebauer believed it to be the outcome of a copyist's mixing up of two problems that happens to make sense mathematically. As we shall see below (page 109), the explanation is quite different.

The archaizing aspect, it should be added, does not dominate completely. Line 12, asking first for the "inscription" of 4 and stating afterwards its IGI, seems to describe the operations on a tablet for rough work that were taught in school (see note 5, page 65, and page 120).

TMS VIII #1

1. The surface 10′. The 4th of the width to the width I have joined, to 3 I have gone … over
2. the length 5′ went beyond. You, 4, of the fourth, as much as width posit. The fourth of 4 take, 1 you see.
3. 1 to 3 go, 3 you see. 4 fourths of the width to 3 join, 7 you see.
4. 7 as much as length posit. 5′ the going-beyond to the to-be-torn-out of the length posit. 7, of the length, to 4, ¿of the width?, raise,

5. 28 you see. 28, of the surfaces, to 10′ the surface raise, 4°40′ you see.
6. 5′, the to-be-torn-out of the length, to four, of the width, raise, 20′ you see. $\frac{1}{2}$ break, 10′ you see. 10′ make hold,
7. 1′40″ you see. 1′40″ to 4°40′ join, 4°41′40″ you see. What is equal? 2°10′ you see.
8. 10′ ¿…? to 2°10′ join, 2°20′ you see. What to 28, of the surfaces, may I posit which 2°20′ gives me?
9. 5′ posit. 5′ to 7 raise, 35′ you see. 5′, the to-be-torn-out of the length, from 35′ tear out,
10. 30′ you see, 30′ the length. 5′ the length to 4 of the width raise, 20′ you see, 20 the length (mistake for width).

In BM 13901 #12 we saw how a problem about squares could be reduced to a rectangle problem. Here, on the contrary, a problem about a rectangle is reduced to a problem about squares.

Translated into symbols, the problem is the following;

$$\tfrac{7}{4}w - \ell = 5' \quad , \quad \sqsubset\!\sqsupset(\ell, w) = 10'$$

("to 3 I have gone" in line 1 means that the "joining" of $\frac{1}{4}w$ in line 1 is repeated thrice). The problem could have been solved in agreement with the methods used in TMS IX #3 (page 57), that is, in the following way:

$$7w - 4\ell = 4 \cdot 5' \quad , \quad \sqsubset\!\sqsupset(\ell, w) = 10'$$
$$7w - 4\ell = 20' \quad , \quad \sqsubset\!\sqsupset(7w, 4\ell) = (7 \cdot 4) \cdot 10' = 28 \cdot 10' = 4°40'$$
$$7w = \sqrt{4°40' + \left(\tfrac{20'}{2}\right)^2} + \tfrac{20'}{2} = 2°20,$$
$$4\ell = \sqrt{4°40' + \left(\tfrac{20'}{2}\right)^2} - \tfrac{20'}{2} = 2$$
$$w = 20' \quad , \quad \ell = 30'.$$

However, once again the calculator shows that he has several strings on his bow, and that he can choose between them as he finds convenient. Here he builds his approach on a square whose side (z) is $\frac{1}{4}$ of the width (see Figure 4.13). In that way, the width will equal 4, understood as $4z$ *(You, 4, of the fourth, as much as width posit)*, and the length prolonged by 5′ will be equal to 7, understood as $7z$ *(7 as much as length posit)*. Line 4 finds that the rectangle with sides $7z$ and $4z$—in other words, the initial rectangle prolonged by 5′—consists of $7 \cdot 4 = 28$

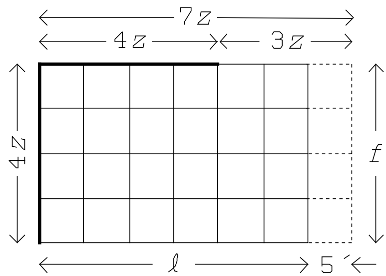

Figure 4.13: The method of TMS VIII #1.

small squares $\square(z)$.[11] These 28 squares exceed the area $10'$ by a certain number of sides $(n \cdot z)$, the determination of which is postponed until later. As usual, indeed, the non-normalized problem

$$28\,\square(z) - n \cdot z = 10'$$

is transformed into

$$\square(28z) - n \cdot (28z) = 28 \cdot 10' = 4°40'.$$

Line 6 finds $n = 4 \cdot 5' = 20'$, and from here onward everything follows the routine, as can be seen on Figure 4.14: $28z$ will be equal to $2°20'$, and z hence to $5'$.[12] Therefore, the length ℓ will be $7 \cdot 5' - 5' = 30'$, and the width w $4 \cdot 5' = 20'$.

YBC 6504 #4

Rev.

11. So much as length over width goes beyond, made encounter, from inside the surface I have torn out:
12. $8'20''$. $20'$ the width, its length what?
13. $20'$ made encounter: $6'40''$ you posit.

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[11]The use of a "raising" multiplication shows that the calculator does not construct a new rectangle but bases his procedure on a subdivision of what is already at hand—see the discussion and dismissal of a possible alternative interpretation of the procedure of BM 13901 #10 in note 5, page 49.

[12]Line 10 speaks of this as *5′ the length*—namely the side of the small square. Some other texts from Susa also speak of the side of a square as its "length."

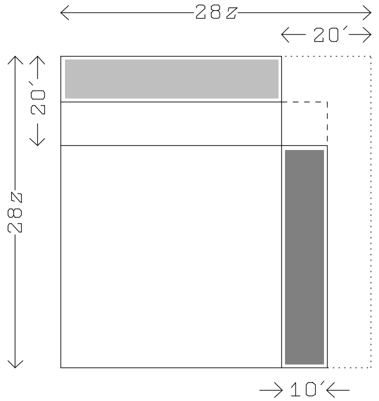

Figure 4.14: Resolution of the normalized equation of TMS VIII #1.

14. 6′40″ to 8′20″ you join: 15′ you posit.
15. By 15′, 30′ is equal. 30′, the length, you posit.

So far, everything we have looked at was mathematically correct, apart from a few calculational and copying errors. But everybody who practises mathematics sometimes also commits errors in the argument; no wonder then that the Babylonians sometimes did so.

The present text offers an example. Translated into symbols, the problem is the following:

$$\sqsubset\!\sqsupset(\ell, w) - \square(\ell - w) = 8'20'' \quad , \quad w = 20' .$$

Astonishingly, the length is found as that which "is equal by" $\sqsubset\!\sqsupset(\ell, w) - \square(\ell - w) + \square(w)$—that is, after a transformation and expressed in symbols, as $\sqrt{(3w - \ell) \cdot \ell}$.

The mistake seems difficult to explain, but inspection of the geometry of the argument reveals its origin (see Figure 4.15). On top the procedure is presented in distorted proportions; we see that the "joining" of $\square(w)$ presupposes that the mutilated rectangle be cut along the dotted line and opened up as a pseudo-gnomon. It is clear that what results from the completion of this configuration is *not* $\square(\ell)$ but instead—if one counts well—$\sqsubset\!\sqsupset(3w - \ell, \ell)$. Below we see the same thing, but now in the proportions of the actual problem, and now the mistake is no longer glaring. Here, $\ell = 30'$ and $w = 20'$, and therefore $\ell - w = w - (\ell - w)$. In con-

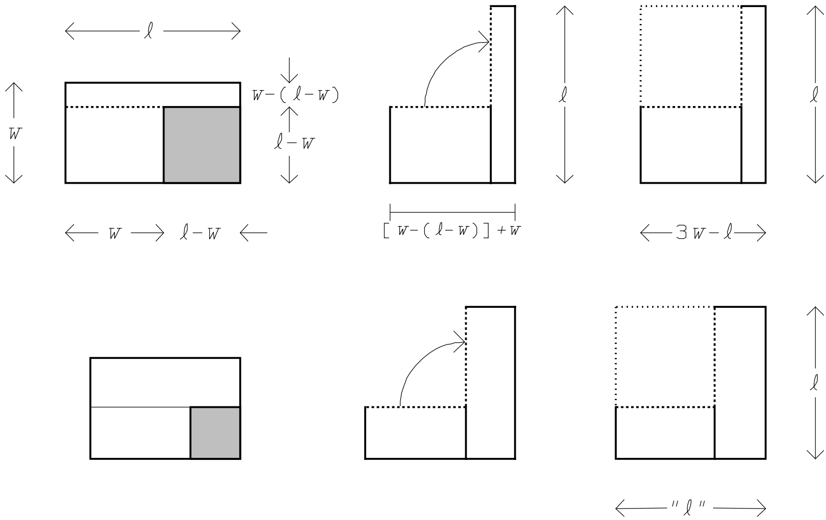

Figure 4.15: The cut-and-paste operations of YBC 6504 #4.

sequence the mutilated rectangle opens up as a true gnomon, and the completed figure corresponds to $\square(\ell)$—but only because $\ell = \frac{3}{2}w$.

This mistake illustrates an important aspect of the "naive" geometry: as is generally the case for geometric demonstrations, scrupulous attention must be paid so that one is not induced into error by what is "immediately" seen. The rarity of such errors is evidence of the high competence of the Old Babylonian calculators and shows that they were almost always able to distinguish the *given magnitudes* of a problem from what more they knew about it.