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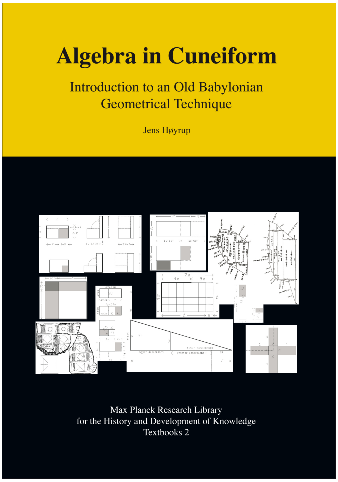
The Fundamental Techniques for the Second Degree

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# Chapter 3
# The Fundamental Techniques for the Second Degree

After these examples of first-degree methods we shall now go on with the principal part of Old Babylonian algebra—postponing once more the precise determination of what "algebra" will mean in a Babylonian context. In the present chapter we shall examine some simple problems, which will allow us to discover the fundamental techniques used by the Old Babylonian scholars. Chapter 4 will take up more complex and subtle matters.

### BM 13901 #1

### Obv. I

1. The surface and my confrontation I have heaped: 45′ is it. 1, the projection,
2. you posit. The moiety of 1 you break, 30′ and 30′ you make hold.
3. 15′ to 45′ you join: by 1, 1 is equal. 30′ which you have made hold
4. from the inside of 1 you tear out: 30′ the confrontation.

This is the problem that was quoted on page 9 in the Assyriologists' "transliteration" and on page 13 in a traditional translation. A translation into modern mathematical symbolism is found on page 12.

Even though we know it well from this point of view, we shall once again examine the text and terminology in detail so as to be able to deal with it in the perspective of its author.

Line 1 states the problem: it deals with a *surface*, here a square, and with its corresponding *confrontation*, that is, the square configuration parametrized by its side, see page 22. It is the appearance of the "confrontation" that tells us that the "surface" is that of a square.

"Surface" and "confrontation" are *heaped*. This addition is the one that must be used when dissimilar magnitudes are involved, here an area (two dimensions) and a side (one dimension). The text tells the sum of the two magnitudes—that is, of their measuring numbers: 45′. If $c$ stands for the side of the square and $\square(c)$ for its area, the problem can thus be expressed in symbols in this way:

$$\square(c) + c = 45' \left(= \tfrac{3}{4}\right).$$

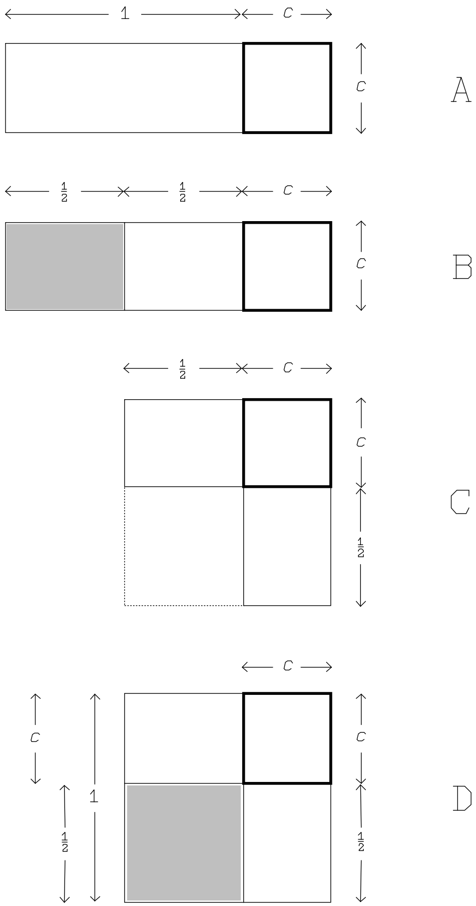


Figure 3.1: The procedure of BM 13901 #1, in slightly distorted proportions.

Figure 3.1 shows the steps of the procedure leading to the solution as they are explained in the text:

A: *1, the projection, you posit.* That means that a rectangle $\sqsubset\sqsupset(c, 1)$ is drawn alongside the square $\Box(c)$. Thereby the sum of a length and an area, absurd in itself, is made geometrically meaningful, namely as a rectangular area $\sqsubset\sqsupset(c, c+1) = \frac{3}{4} = 45'$. This geometric interpretation explains the appearance of the "projection," since the rectangle $\sqsubset\sqsupset(c,1)$ "projects" from the square as a projection protruding from a building. We remember (see page 15) that the word was originally translated as "unity" or "coefficient" simply because the translators did not understand how a number 1 could "project."

B: *The moiety of 1 you break.* The "projection" with adjacent rectangle $\sqsubset\!\sqsupset(c,1)$ is "broken" into two "natural" halves.

C: *30′ and 30′ you make hold.* The outer half of the projection (shaded in grey) is moved around in such a way that its two parts (each of length 30′) "hold" the square with dotted border below to the left. This cut-and-paste procedure has thus allowed us to transform the rectangle $\sqsubset\!\sqsupset(c,c+1)$ into a "gnomon," a square from which a smaller square is lacking in a corner.

D: *15′ to 45′ you join: 1.* 15′ is the area of the square held by the two halves (30′ and 30′), and 45′ that of the gnomon. As we remember from page 18, to "join" one magnitude to another one is an enlargement of the latter and only possible if both are concrete and of the same kind, for instance areas. We thus "join" the missing square, completing in this way the gnomon in order to get a new square. The area of the completed square will be $45' + 15' = 1$.

*by 1, 1 is equal.* In general, the phrase "by $Q$, $s$ is equal" means (see page 23) that the area $Q$ laid out as a square has $s$ as one of its equal sides (in arithmetical language, $s = \sqrt{Q}$). In the present case, the text thus tells us that the side of the completed square is 1, as indicated in D immediately to the left of the square.

*30′ which you have made hold from the inside of 1 you tear out.* In order to find the side $c$ of the original square we must now remove that piece of length $\frac{1}{2} = 30'$ which was added to it below. To "tear out" $a$ from $H$, as we have seen on page 18, is the inverse operation of a "joining," a concrete elimination which presupposes that $a$ is actually a part of $H$. As observed above (page 15), the phrase "from the inside" was omitted from the early translations, being meaningless as long as everything was supposed to deal with abstract numbers. If instead the number 1 represents a segment, the phrase does make sense.

*30′ the confrontation*. Removing from 1 the segment $\frac{1}{2} = 30'$ which was added, we get the initial side $c$, the "confrontation," which is hence equal to $1 - 30' = 30' = \frac{1}{2}$ (extreme left in *D*).

That solves the problem. In this geometric interpretation, not only the numbers are explained but also the words and explanations used in the text.

The new translation calls for some observation. We take note that no explicit argument is given that the cut-and-paste procedure leads to a correct result. On the other hand it is intuitively clear that it must be so. We may speak of a "naive" approach—while keeping in mind that *our* normal way to operate on equations, for instance in the example solving the same problem on page 12, is no less naive. Just as the Old Babylonian calculator we proceed from step to step without giving any explicit proof that the operations we make are justified, "seeing" merely that they are appropriate.

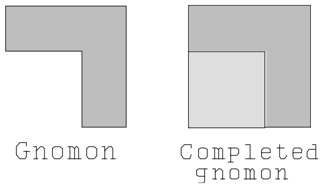


Figure 3.2

The essential stratagem of the Old Babylonian method is the completion of the gnomon as shown in Figure 3.2. This stratagem is called a "quadratic completion"; the same term is used about the corresponding step in our solution by means of symbols:

$$
\begin{aligned}
x^2+1\cdot x=\tfrac{3}{4} \quad &\Leftrightarrow \quad x^2+1\cdot x+\left(\tfrac{1}{2}\right)^2=\tfrac{3}{4}+\left(\tfrac{1}{2}\right)^2 \\
&\Leftrightarrow \quad x^2+1\cdot x+\left(\tfrac{1}{2}\right)^2=\tfrac{3}{4}+\tfrac{1}{4}=1 \\
&\Leftrightarrow \quad \left(x+\tfrac{1}{2}\right)^2=1.
\end{aligned}
$$

However, the name seems to apply even better to the geometric procedure.

It is obvious that a negative solution would make no sense in this concrete interpretation. Old Babylonian algebra was based on tangible quantities even in cases where its problems were not really practical. No length (nor surface, volume or weight) could be negative. The only idea found in the Old Babylonian texts that approaches negativity is that a magnitude can be *subtractive*, that is, pre-determined to be torn out. We have encountered such magnitudes in the text TMS XVI #1 (lines 3 and 4—see page 27) as well as TMS VII #2 (line 35, the "to-be-torn-out of the width"—see page 34). In line 25 of the latter text we also observe that the Babylonians did not consider the outcome of a subtraction of $20'$ from $20'$ as a *number* but, literally, as *something not worth speaking of*.

Certain general expositions of the history of mathematics claim that the Babylonians did know of negative numbers. This is a legend based on sloppy reading. As mentioned, some texts state for reasons of style not that a magnitude *A* exceeds another one by the amount *d* but that B falls short of *A* by *d*; we shall encounter an example in BM 13901 #10, see note 4, page 46. In his *mathematical commentaries* Neugebauer expressed these as respectively $A-B=d$ and

$B - A = -d$ ($A = B + d$ and $B = A - d$ would have been closer to the ancient texts, but even Neugebauer had his reasons of style). In this way, mathematicians who only read the translations into formulas and not the explanations of the meaning of these (and certainly not the translated texts) found their "Babylonian" negative numbers.

As the French Orientalist Léon Rodet wrote in 1881 when criticizing modernizing interpretations of an ancient Egyptian mathematical papyrus:

> For studying the history of a science, just as when one wants to obtain something, 'it is better to have business with God than with his saints'.[1]

**BM 13901 #2**

**Obv. I**

5. My confrontation inside the surface I have torn out: 14ˋ30 is it. 1, the projection,
6. you posit. The moiety of 1 you break, 30′ and 30′ you make hold,
7. 15′ to 14ˋ30 you join: by 14ˋ30°15′, 29°30′ is equal.
8. 30′ which you have made hold to 29°30′ you join: 30 the confrontation.

This problem, on a tablet which contains in total 24 problems of increasing sophistication dealing with one or more squares, follows immediately after the one we have just examined.

From the Old Babylonian point of view as well as ours, it is its "natural" counterpart. Where the preceding one "joins," this one "tears out." The basic part of the procedure is identical: the transformation of a rectangle into a gnomon, followed by a quadratic complement.

Initially the problem is stated (line 5): *My confrontation inside the surface I have torn out: 14ˋ30 is it.* Once again the problem thus concerns a square area and side, but this time the "confrontation" $c$ is "torn out."

To "tear out" is a concrete subtraction by removal, the inverse of the "joining" operation, used only when that which is "torn out" is part of that magnitude from which it is "torn out."[2] The "confrontation" $c$ is thus seen as part of (the inside of) the area. Figure 3.3, A shows how this is possible: the "confrontation" $c$ is provided with a width (a "projection") 1 and thereby changed into a rectangle ⊏⊐($c$,1), located inside the square. This rectangle (shaded in dark grey) must thus

[1] Léon Rodet, *Journal asiatique*, septième série **18**, p. 205.

[2] The inverse of the "heaping" operation, on the other hand, is no subtraction at all but a *separation into constitutive elements*. See note 3, page 99.

be "torn out"; what remains after we have eliminated $\sqsubset\!\sqsupset(c,1)$ from $\square(c)$ should be 14`30. In modern symbols, the problem corresponds to

$$\square(c) - c = 14`30.$$

Once more, we are left with a rectangle for which we know the area (14`30) and the difference between the length ($c$) and the width ($c - 1$)—and once more, this difference amounts to 1, namely the "projection."

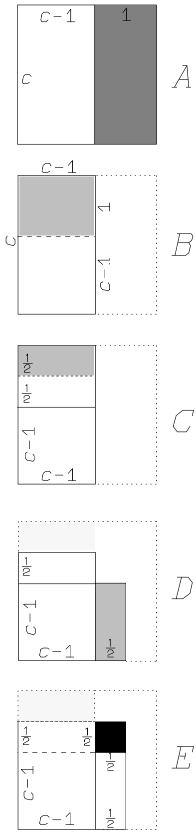


Figure 3.3: The procedure of BM 13901 #2.

*1, the projection, you posit.* In Figure 3.3, B, the rectangle $\sqsubset\!\sqsupset(c, c-1)$ is composed of a (white) square and a (shaded) "excess" rectangle whose width is the projection 1.

*The moiety of 1 you break.* The excess rectangle, presented by its width 1, is divided into two "moieties"; the one which is detached is shaded in Figure 3.3, C.

Cutting and pasting this rectangle as seen in Figure 3.3, D we once again get a gnomon with the same area as the rectangle $\sqsubset\!\sqsupset(c, c-1)$, that is, equal to 14`30.

*30′ and 30′ you make hold, 15′.* The gnomon is completed with the small square (black in Figure 3.3, E) which is "held" by the two moieties. The area of this completing square equals $30' \times 30' = 15'$.

Next, the area of the completed square and its side are found: *15′ to 14`30 you join: by 14`30°15′, 29°30′ is equal.*

Putting back the "moiety" which was moved around, we find the side of the initial square, which turns out to be 29°30′+30′ = 30: *30′ which you have made hold to 29°30′ you join: 30 the confrontation*.

We notice that this time the "confrontation" of the square is 30, not 30′. The reason is simple and compelling: unless $c$ is larger than 1, the area will be smaller than the side, and we would have to "tear out" more than is available, which evidently cannot be done. As already explained, the Babylonians were familiar with "subtractive magnitudes," that is, magnitudes that are predetermined to be "torn out"; but nothing in their mathematical thought corresponded to our negative numbers.

We also notice that the pair (14`30°15′, 29°30′) does not appear in the table of squares and square roots (see page 23); the problem is thus constructed backwards from a known solution.

## YBC 6967

### Obv.

1. The *igibûm* over the *igûm*, 7 it goes beyond
2. *igûm* and *igibûm* what?
3. You, 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30′;
6. 3°30′ together with 3°30′
7. make hold: 12°15′.
8. To 12°15′ which comes up for you
9. 1` the surface join: 1`12°15′.
10. The equal of 1`12°15′ what? 8°30′.
11. 8°30′ and 8°30′, its counterpart, lay down.

**Rev.**

1. 3°30′, the made-hold,
2. from one tear out,
3. to one join.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.

Second-degree problems dealing with rectangles are more copious than those about squares. Two problem types belong to this category; others, more complex, can be reduced to these basic types. In one of these, the area and the sum of the sides are known; in the other, the area and their difference are given.

The above exercise belongs to the latter type—if we neglect the fact that it does not deal with a rectangle at all but with a pair of numbers belonging together in the table of reciprocals (see page 20 and Figure 1.2). *Igûm* is the Babylonian pronunciation of Sumerian IGI, and *igibûm* that of IGI.BI, "its IGI" (the relation between the two is indeed symmetric: if 10′ is IGI 6, then 6 is IGI 10′).

One might expect the product of *igûm* and *igibûm* to be 1; in the present problem, however, this is not the case, here the product is supposed to be 1`, that is, 60. The two numbers are represented by the sides of a rectangle of area 1` (see line Obv. 9); the situation is depicted in Figure 3.4, A. Once more we thus have to do with a rectangle with known area and known difference between the length and the width, respectively 1` and 7.

It is important to notice that here the "fundamental representation" (the measurable geometric quantities) serves to represent magnitudes of a different kind: the two numbers *igûm* and *igibûm*. In our algebra, the situation is the inverse: *our* fundamental representation is provided by the realm of abstract numbers, which serves to represent magnitudes of other kinds: prices, weights, speeds, distances, etc. (see page 16).

As in the two analogous cases that precede, the rectangle is transformed into a gnomon, and as usually the gnomon is completed as a square "held" by the two "moieties" of the excess (lines Obv. 3–10). The procedure can be followed on the Figures 3.4, B and 3.4, C.

The next steps are remarkable. The "moiety" that was detached and moved around (the "made-hold," that is, that which was "made hold" the complementary square) in the formation of the gnomon is put back into place. Since it is *the same* piece which is concerned it must in principle be available before it can be "joined." That has two consequences. Firstly, the "equal" 8°30′ must be "laid down"[3] twice, as we see in Figure 3.4, D: in this way, the piece can be "torn out"

[3]The verb in question (*nadûm*) has a broad spectrum of meanings. Among these are "to draw" or "to write" (on a tablet) (by the way, the word *lapātum*, translated "to inscribe," has the same two

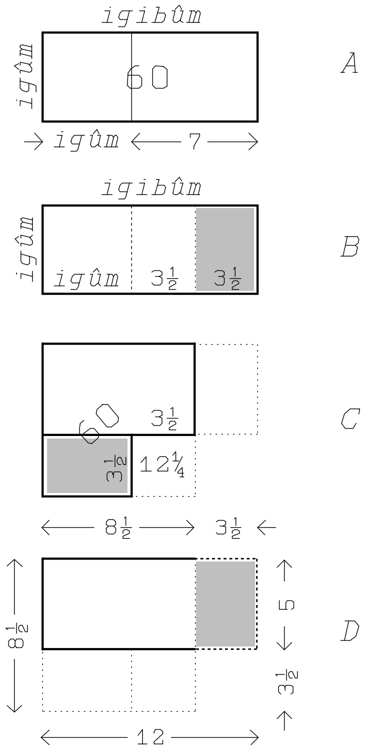


Figure 3.4: The procedure of YBC 6967.

from one (leaving the width *igûm*) and "joined" to the other (giving the length *igibûm*). Secondly, "tearing-out" must precede "joining" (lines Rev. 1–3), even though the Babylonians (as we) would normally prefer to add before subtracting—cf. BM 13901 #1–2: the first problem adds the side, the second subtracts: *3°30′, the made-hold, from one tear out, to one join.*

In BM 13901 #1 and #2, the complement was "joined" to the gnomon, here it is the gnomon that is "joined." Since both remain in place, either is possible. When 3°30′ is joined to 8°30′ in the construction of the *igibûm*, this is not the case: if one magnitude stays in place and the other is displaced it is always the

meanings). Since what is "laid down" is a numerical value, the latter interpretation could seem to be preferable—but since geometrical entities were regularly identified by means of their numerical measure, this conclusion is not compulsory.

latter that is "joined." Differently from our addition and the "heaping" of the Babylonians, "joining" is no symmetric operation.

**BM 13901 #10**

**Obv. II**

11. The surfaces of my two confrontations I have heaped: 21°15′.
12. Confrontation (compared) to confrontation, the seventh it has become smaller.
13. 7 and 6 you inscribe. 7 and 7 you make hold, 49.
14. 6 and 6 you make hold, 36 and 49 you heap:
15. 1`25. IGI 1`25 is not detached. What to 1`25
16. may I posit which 21°15′ gives me? By 15′, 30′ is equal.
17. 30′ to 7 you raise: 3°30′ the first confrontation.
18. 30′ to 6 you raise: 3 the second confrontation.

We now return to the tablet containing a collection of problems about squares, looking at one of the simplest problems about two squares. Lines 11 and 12 contain the statement: the sum of the two areas is told to be 21°15′, and we are told that the second "confrontation" falls short of the first by one seventh.[4] In symbols, if the two sides are designated respectively $c_1$ and $c_2$:

$$\square(c_1)+\square(c_2)=21°15' \quad , \quad c_2=c_1-\tfrac{1}{7}c_1 .$$

Formulated differently, the ratio between the two sides is as 7 to 6. This is the basis for a solution based on a "false position" (see page 32). Lines 13 and 14 prescribe the construction of two "model squares" with sides 7 and 6 (making these sides "hold," see Figure 3.5), and finds that their total area will be 49+36 = 1`25. According to the statement, however, the total *should* be 21°15′; therefore, the area must be reduced by a factor 21°15′/ 1`25. Now 1`25 is no "regular" number (see page 21)—that is, it has no IGI: *IGI 1`25 is not detached*. We must thus draw the quotient "from the sleeves"—as done in lines 15–16, where it is said to be 15′ (that is, $\frac{1}{4}$). However, if the area is reduced by a factor 15′, then the corresponding sides must be reduced by a factor 30′: *By 15′, 30′ is equal.* It remains finally (lines 17 and 18) to "raise" 7 and 6 to 30′.

[4]Here we see one of the stylistic reasons that would lead to a formulation in terms of falling-short instead of excess. It might as well have been said that one side exceeds the other by one sixth, but in the "multiplicative-partitive" domain the Babylonians gave special status to the numbers 4, 7, 11, 13, 14 and 17. In the next problem on the tablet, one "confrontation" is stated to exceed the other by one seventh, while it would be just as possible to say that the second falls short of the first by one eighth.

The first "confrontation" thus turns out to be $7 \cdot 30' = 3°30'$, and the second $6 \cdot 30' = 3$.[5]

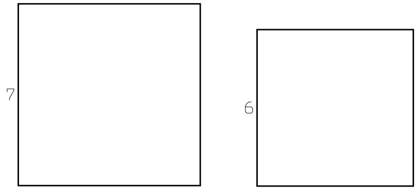


Figure 3.5: The two squares of BM 13901 #10.

**BM 13901 #14**

**Obv. II**

44. The surfaces of my two confrontations I have heaped: $25'25''$.
45. The confrontation, two-thirds of the confrontation and $5'$, NINDAN.
46. 1 and $40'$ and $5'$ over-going $40'$ you inscribe.
47. $5'$ and $5'$ you make hold, $25''$ inside $25'25''$ you tear out:

**Rev. I**

1. $25'$ you inscribe. 1 and 1 you make hold, 1. $40'$ and $40'$ you make hold,
2. $26'40''$ to 1 you join: $1°26'40''$ to $25'$ you raise:
3. $36'6''40'''$ you inscribe. $5'$ to $40'$ you raise: $3'20''$
4. and $3'20''$ you make hold, $11''6'''40''''$ to $36'6''40'''$ you join:
5. by $36'17''46'''40''''$, $46'40''$ is equal. $3'20''$ which you have made hold
6. inside $46'40''$ you tear out: $43'20''$ you inscribe.
7. IGI $1°26'40''$ is not detached. What to $1°26'40''$
8. may I posit which $43'20''$ gives me? $30'$ its *bandûm*.
9. $30'$ to 1 you raise: $30'$ the first confrontation.
10. $30'$ to $40'$ you raise: $20'$, and $5'$ you join:
11. $25'$ the second confrontation.

[5] One might believe the underlying idea to be slightly different, and suppose that the original squares are subdivided into 7×7 respectively 6×6 smaller squares, of which the total number would be 1`25, each thus having an area equal to $\frac{21°15'}{1`25} = 15'$ and a side of $30'$. However, this interpretation is ruled out by the use of the operation "to make hold": Indeed, the initial squares are already there, and there is thus no need to construct them (in TMS VIII #1 we shall encounter a subdivision into smaller squares, and there their number is indeed found by "raising"—see page 78).

Even this problem deals with two squares (lines Obv. II.44–45).[6] The somewhat obscure formulation in line 45 means that the second "confrontation" equals two-thirds of the first, with additional 5′ NINDAN. If $c_1$ and $c_2$ stands for the two "confrontations," line 44 informs us that the sum of the areas is $\square(c_1) + \square(c_2) = 25'25''$, while line 45 states that $c_2 = 40' \cdot c_1 + 5'$.

This problem cannot be solved by means of a simple false position in which a hypothetical number is provisionally assumed as the value of the unknown—that only works for homogeneous problems.[7] The numbers 1 and 40′ in line 46 show us the way that is actually chosen: $c_1$ and $c_2$ are expressed in terms of a *new magnitude*, which we may call $c$:

$$c_1 = 1 \cdot c \quad , \quad c_2 = 40' \cdot c + 5'.$$

That corresponds to Figure 3.6. It shows how the problem is reduced to a simpler one dealing with a single square $\square(c)$. It is clear that the area of the first of the two original squares ($\square(c_1)$) equals $(1 \times 1)\square(c)$, but that calculation has to wait until line Rev. I.1. The text begins by considering $\square(c_2)$, which is more complicated and gives rise to several contributions. First, the square $\square(5')$ in the lower right corner: *5′ and 5′ you make hold,* 25″. This contribution is eliminated from the sum 25′25″ of the two areas: 25″ *inside* 25′25″ *you tear out: 25′ you inscribe*. The 25′ that remains must now be explained in terms of the area and the side of the new square $\square(c)$.

$\square(c_1)$, as already said, is $1 \times 1 = 1$ times the area $\square(c)$: *1 and 1 you make hold, 1.*[8] After elimination of the corner $5' \times 5'$ remains of $\square(c_2)$, on one hand, a square $\square(40'c)$, on the other, two "wings" to which we shall return imminently. The area of the square $\square(40'c)$ is $(40' \times 40')\square(c) = 26'40''\square(c)$: *40′ and 40′ you make hold, 26′40″*. In total we thus have $1 + 26'40'' = 1°26'40''$ times the square area $\square(c)$: *26′40″ to 1 you join: 1°26′40″*.

[6] This part of the tablet is heavily damaged. However, #24 of the same tablet, dealing with three squares but otherwise strictly parallel, allows an unquestionable reconstruction.

[7] In a simple false position, indeed, the provisionally assumed number has to be reduced by a factor corresponding to the error that is found; but if we reduce values assumed for $c_1$ and $c_2$ with a certain factor—say, $\frac{1}{5}$—then the additional 5′ would be reduced by the same factor, that is, to 1′. After reduction we would therefore have $c_2 = \frac{2}{3}c_1 + 1'$.

[8] This meticulous calculation shows that the author thinks of a *new* square, and does not express $\square(c_2)$ in terms of $\square(c_1)$ and $c_1$.

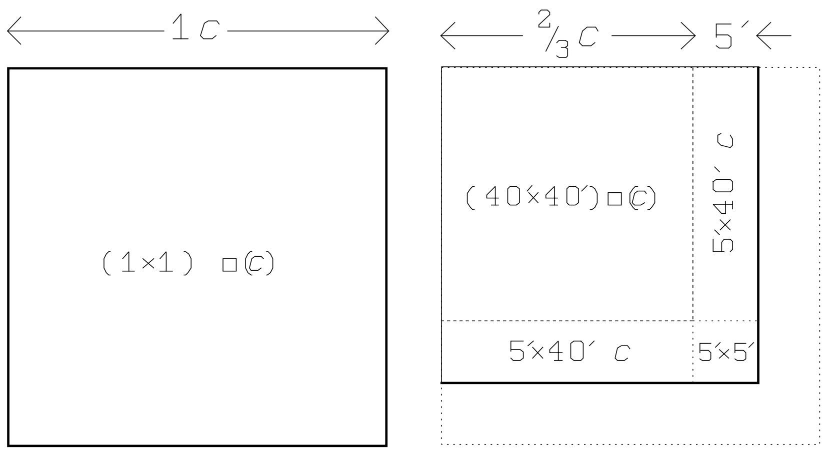


Figure 3.6: The two squares of BM 13901 #14.

Each "wing" is a rectangle $\sqsubset\!\sqsupset(5', 40'c)$, whose area can be written $5' \cdot 40'c = 3'20''c$: *5′ to 40′ you raise: 3′20″*. All in all we thus have the equation

$$1°26'40''\square(c) + 2 \cdot 3'20''c = 25'.$$

This equation confronts us with a problem which the Old Babylonian author has already foreseen in line Rev. I.2, and which has caused him to postpone until later the calculation of the wings. In modern terms, the equation is not "normalized," that is, the coefficient of the second-degree term differs from 1. The Old Babylonian calculator might correspondingly have explained it by stating in the terminology of TMS XVI that "as much as (there is) of surfaces" is not one—see the left part of Figure 3.7, where we have a sum of $\alpha$ square areas (the white rectangle $\sqsubset\!\sqsupset(c, \alpha c)$) and $\beta$ sides, that is, the shaded rectangle $\sqsubset\!\sqsupset(c, \beta)$, corresponding to the equation

$$\alpha\square(c) + \beta c = \Sigma$$

(in the actual case, $\alpha = 1°26'40''$, $\beta = 2 \cdot 3'20''$, $\Sigma = 25'$). This prevents us from using directly our familiar cut-and-paste procedure. "Breaking" $\beta$ and making the two "moieties" "hold" would not give us a gnomon.

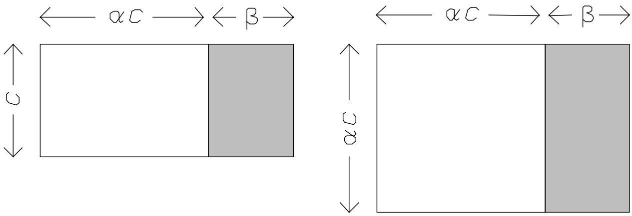


Figure 3.7: Transformation of the problem $\alpha\square(c) + \beta c = \Sigma$.

The Babylonians got around the difficulty by means of a device shown in the right-hand side of figure 3.7: the scale of the configuration is changed in the vertical direction, in such a way that the vertical side becomes $\alpha c$ instead of $c$; in consequence the sum of the two areas is no longer $\Sigma\,(= 25')$ but $\alpha\Sigma(= 1°26'40'' \cdot 25' = 36'6''40''')$: $1°26'40''$ *to 25′ you raise:* $36'6''40'''$ you inscribe. As we see, the number $\beta$ of sides is not changed in the operation, only the value of the side, namely from $c$ into $\alpha c$.[9]

In modern symbolic language, this transformation corresponds to a multiplication of the two sides of the equation

$$\alpha c^2 + \beta c = \Sigma$$

by $\alpha$, which gives us a normalized equation with the unknown $\alpha c$:

$$(\alpha c)^2 + \beta \cdot (\alpha c) = \alpha\Sigma,$$

an equation of the type we have encountered in BM 13901 #1. We have hence arrived to a point where we can apply the habitual method: "breaking" the shaded rectangle and make the two resulting "moieties" "hold" a quadratic complement (see Figure 3.8); the outer "moiety" is lightly shaded in its original position and more heavily in the position to which it is brought). Now, and only now, does the calculator need to know the number of sides in the shaded rectangle of Figure 3.7 (that is, to determine $\beta$). As already said, each "wing" contributes $5'40'' = 3'20''$ sides. If the calculator had worked mechanically, according to fixed algorithms,

[9]This device was used constantly in the solution of non-normalized problems, and there is no reason to suppose that the Babylonians needed a specific representation similar to Figure 3.7. They might imagine that the *measuring scale* was changed in one direction—we know from other texts that their diagrams could be very rough, mere structure diagrams—nothing more than was required in order to guide thought. All they needed was thus to multiply the sum $\Sigma$ by $\alpha$, and that they could (and like here, would) do before calculating $\beta$.

he would now have multiplied by 2 in order to find $\beta$. But he does not! He knows indeed that the two wings constitute the excess that has to be "broken" into two "moieties." He therefore directly makes $3'20''$ and $3'20''$ "hold," which produces the quadratic complement, and "joins" the resulting area $11''6'''40''''$ to that of the gnomon $36'6''40'''$: $3'20''$ *and* $3'20''$ *you make hold,* $11''6'''40''''$ *to* $36'6''40'''$ *you join: [...]* $36'17''46'''40''''$.

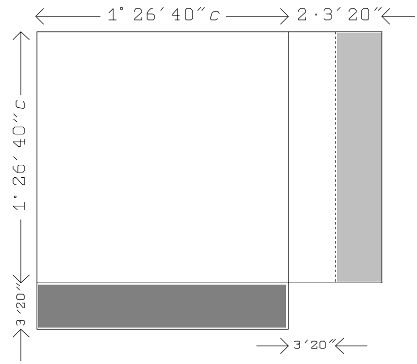


Figure 3.8: BM 13901 #14, the normalized problem.

$36'17''46'''40''''$ is thus the area of the completed square, and its side $\sqrt{36'17''46'''40''''} = 46'40''$: *by* $36'17''46'''40''''$, $46'40''$ *is equal*. This number represents $1°26'40''\cdot c+3'20''$; therefore, $1°26'40''c$ is $46'40''-3'20'' = 43'20''$: $3'20''$ *which you have made hold inside* $46'40''$ *you tear out:* $43'20''$ *you inscribe.* Next, we must find the value of $c$. $1°26'40''$ is an irregular number, and the quotient $46'40''/1°26'40''$ is given directly as $30'$:[10] *IGI* $1°26'40''$ *is not detached. What to* $1°26'40''$ *may I posit which* $43'20''$ *gives me?* $30'$ *its bandûm*.

In the end, $c_1$ and $c_2$ are determined, $c_1 = 1\cdot c = 30'$, $c_2 = 40'\cdot c+5' = 25'$:[11] *30′ to 1 you raise: 30′ the first confrontation. 30′ to 40′ you raise: 20′, and 5′ you join: 25′ the second confrontation.* The problem is solved.

[10] The quotient is called BA.AN.DA. This Sumerian term could mean "that which is put at the side," which would correspond to way multiplications were performed on a tablet for rough work, cf. note 11, page 21.

[11] That the value of $c_1$ is calculated as $1\cdot c$ and not directly identified with $c$ confirms that we have been working with a *new* side $c$.

## TMS IX #1 and #2

### #1

1. The surface and 1 length I have heaped, 40′. ¿30, the length,? 20′ the width.
2. As 1 length to 10′ the surface, has been joined,
3. or 1 (as) base to 20′, the width, has been joined,
4. or 1°20′ ¿is posited? to the width which 40′ together with the length ¿holds?
5. or 1°20′ toge⟨ther⟩ with 30′ the length holds, 40′ (is) its name.
6. Since so, to 20′ the width, which is said to you,
7. 1 is joined: 1°20′ you see. Out from here
8. you ask. 40′ the surface, 1°20′ the width, the length what?
9. 30′ the length. Thus the procedure.

### #2

10. Surface, length, and width I have heaped, 1. By the Akkadian (method).
11. 1 to the length join. 1 to the width join. Since 1 to the length is joined,
12. 1 to the width is joined, 1 and 1 make hold, 1 you see.
13. 1 to the heap of length, width and surface join, 2 you see.
14. To 20′ the width, 1 join, 1°20′. To 30′ the length, 1 join, 1°30′.
15. ¿Since? a surface, that of 1°20′ the width, that of 1°30′ the length,
16. ¿the length together with? the width, are made hold, what is its name?
17. 2 the surface.
18. Thus the Akkadian (method).

As TMS XVI #1, sections #1 and #2 of the present text solve no problem.[12] Instead they offer a pedagogical explanation of the meaning to ascribe to the addition of areas and lines, and of the operations used to treat second-degree problems. Sections #1 and #2 set out two different situations. In #1, we are told the sum of the area and the length of a rectangle; in #2, the sum of area, length and width is given. #3 (which will be dealt with in the next chapter) is then a genuine problem that is stated and solved in agreement with the methods taught in #1 and #2 and in TMS XVI #1.

Figure (3.9) is drawn in agreement with the text of #1, in which the sum of a rectangular area and the corresponding length is known. In parallel with our symbolic transformation

$$\ell \cdot w + \ell = \ell \cdot w + \ell \cdot 1 = \ell \cdot (w + 1),$$

[12]The tablet is rather damaged; as we remember, passages in ¿…? are reconstructions that render the meaning (which can be derived from the context) but not necessarily the exact words of the original.

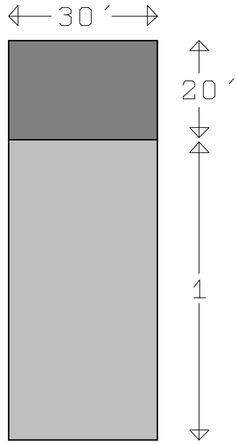


Figure 3.9: TMS IX, #1.

the width is extended by a "base."[13] That leads to a whole sequence of explanations, mutually dependent and linked by "or … or … or," curiously similar to how we speak about the transformations of an equation, for example

$$\text{“}2a^2 - 4 = 4, \quad \textit{or} \quad 2a^2 = 4 + 4, \quad \textit{or} \quad a^2 = 4, \quad \textit{or} \quad a = \pm\sqrt{4} = \pm 2\text{”}.$$

Line 2 speaks of the "surface" as 10′. This shows that the student is once more supposed to know that the discussion deals with the rectangle ⊏⊐(30′,20′). The tablet is broken, for which reason we cannot know whether the length was stated explicitly, but the quotation in line 6 shows that the width was.

In the end, lines 7–9 shows how to find the length once the width is known together with the sum of area and length (by means of a division that remains implicit).

#2 teaches how to confront a more complex situation; now the sum of the area and both sides is given (see Figure 3.10). Both length and width are prolonged by 1; that produces two rectangles ⊏⊐($\ell$, 1) and ⊏⊐($w$, 1), whose areas, respectively, are the length and the width. But it also produces an empty square corner ⊏⊐(1,1). When it is filled we have a larger rectangle of length

[13]The word KI.GUB.GUB is a composite Sumerian term that is not known from elsewhere and which could be an *ad hoc* construction. It appears to designate something stably placed on the ground.

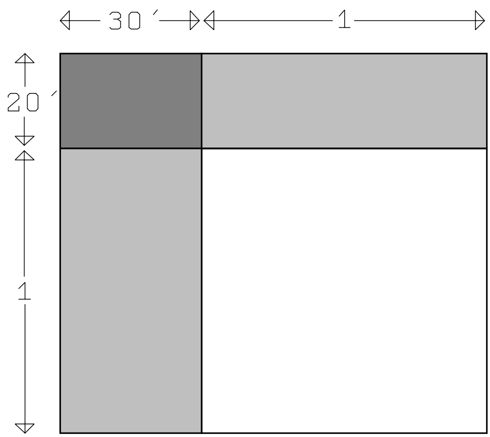


Figure 3.10: TMS IX, #2.

$\ell + 1$ (= 1°30′), width $w + 1$ (= 1°20′) and area $1 + 1 = 2$; a check confirms that the rectangle "held" by these two sides is effectively of area 2.

This method has a name, which is very rare in Old Babylonian mathematics (or at least in its written traces). It is called "the Akkadian (method)." "Akkadian" is the common designation of the language whose main dialects are Babylonian and Assyrian (see the box "Rudiments of general history"), and also of the major non-Sumerian component of the population during the third millennium; there is evidence (part of which is constituted by the present text) that the Old Babylonian scribe school took inspiration for its "algebra" from the practice of an Akkadian profession of surveyors (we shall discuss this topic on page 108). The "Akkadian" method is indeed nothing but a *quadratic completion* albeit a slightly untypical variant, that is, the basic tool for the solution of all mixed second-degree problems (be they geometric or, as with us, expressed in number algebra); and it is precisely this basic tool that is characterized as the "Akkadian (method)."