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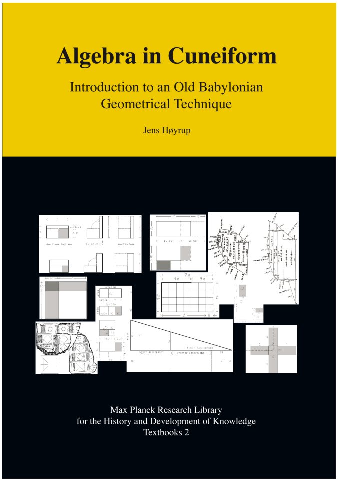
Jens Høyrup:

Techniques for the First Degree



In: Jens Høyrup: *Algebra in Cuneiform : Introduction to an Old Babylonian Geometrical Technique*
Online version at `http://mprl-series.mpg.de/textbooks/2/`

ISBN 978-3-945561-15-7
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Printed and distributed by:
PRO BUSINESS digital printing Deutschland GmbH, Berlin
`http://www.book-on-demand.de/shop/15336`

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at `http://dnb.d-nb.de`

Chapter 2
Techniques for the First Degree

Our main topic will be the Old Babylonian treatment of second-degree equations.[1] However, the solution of second-degree equations or equation systems often asks for first-degree manipulations, for which reason it will be useful to start with a text which explains how first-degree equations are transformed and solved.

TMS XVI #1

1. The 4th of the width, from the length and the width I have torn out, 45′. You, 45′
2. to 4 raise, 3 you see. 3, what is that? 4 and 1 posit,
3. 50′ and 5′, to tear out, posit. 5′ to 4 raise, 1 width. 20′ to 4 raise,
4. 1°20′ you ⟨see⟩,[2] 4 widths. 30′ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20′, 1 width, to tear out,
5. from 1°20′, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, heap, 3 you see.
6. IGI 4 detach, 15′ you see. 15′ to 2, lengths, raise, 30′ you ⟨see⟩, 30′ the length.
7. 15′ to 1 raise, 15′ the contribution of the width. 30′ and 15′ hold.
8. Since "The 4th of the width, to tear out," it is said to you, from 4, 1 tear out, 3 you see.
9. IGI 4 de ⟨tach⟩, 15′ you see, 15′ to 3 raise, 45′ you ⟨see⟩, 45′ as much as (there is) of widths.
10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1′ raise, 20′ you see.

[1] As in the case of "algebra" we shall pretend for the moment to know what an "equation" is. Analysis of the present text will soon allow us to understand in which sense the Old Babylonian problems can be understood as equations.

[2] "you ⟨see⟩" translates *ta*-⟨*mar*⟩. The scribe thus does not omit a word, he uses the first syllable (which happens to carry the information about the grammatical person) as a logogram for the whole word. This is very common in the texts from Susa, and illustrates that the use of logograms is linked to the textual genre: only in mathematical texts can we be reasonably sure that no other verbs beginning with the syllable *ta* will be present in this position.

11. 20′ to 45′ raise, 15′ you see. 15′ from $^{30}{}_{15'}$ tear out,
12. 30′ you see, 30′ the length.

This text differs in character from the immense majority of Old Babylonian mathematical texts: it does not state a problem, and it solves none. Instead, it gives a didactic explanation of the concepts and procedures that serve to understand and reduce a certain often occurring equation type.

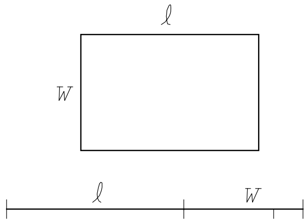

Figure 2.1: The geometry of TMS XVI #1.

Even though many of the terms that appear in the translation were already explained in the section "A new interpretation," it may be useful to go through the text word for word.

Line 1 formulates an equation: *The 4th of the width, from the length and the width I have torn out, 45′.*

The equation thus concerns a length and a width. That tells us that the object is a rectangle—from the Old Babylonian point of view, the rectangle is the simplest figure determined by a length and a width alone.[3] Concerning the number notation, see the box "The sexagesimal system," page 14. If ℓ is the length and w the width, we may express the equation in symbols in this way:

$$(\ell + w) - \tfrac{1}{4}w = 45'.$$

Something, however, is lost in this translation. Indeed, *the length and the width* is a condensed expression for a "heaping," the symmetric addition of two magnitudes (or their measuring numbers; see page 18). The length is thus not prolonged

[3] A right triangle is certainly also determined by a length and a width (the legs of the right angle), and these two magnitudes suffice to determine it (the third side, if it appears, may be "the long length"). But a triangle is always introduced as such. If it is not practically right, the text will give a sketch.

The word "practically" should be taken note of. The Babylonians had no concept of the angle as a measurable quantity—thus, nothing corresponding to our "angle of 78°." But they distinguished clearly "good" from "bad" angles—we may use the pun that the opposite of a *right angle* was a *wrong angle*. A *right angle* is one whose legs determine an area—be it the legs of the right angle in a right triangle, the sides of a rectangle, or the height and the average base of a right trapezium.

by the width, the two magnitudes are combined on an equal footing, independently of the rectangle. The sole role of the rectangle is to put its dimensions at disposal as unknown magnitudes (see Figure 2.1).

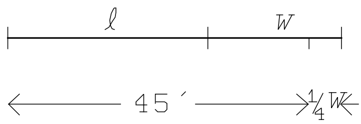

Figure 2.2: "The equation" of TMS XVI #1.

Once the length and the width have been "heaped," it is possible to "tear out" $\frac{1}{4}w$, since this entity is a part of the width and hence also of the total. To "tear out," as we remember, is the inverse operation of "joining," and thus the removal of a magnitude from another one of which it is a part (see Figure 2.2).

Line 1 shows the nature of a Babylonian equation: a combination of measurable magnitudes (often, as here, geometric magnitudes), for which the total is given. Alternatively the text states that the measure of one combination is equal to that of another one, or by how much one exceeds the other. That is not exactly the type of equation which is taught in present-day school mathematics, which normally deals with pure numbers—but it is quite similar to the equations manipulated by engineers, physicists or economists. To speak of "equations" in the Babylonian context is thus not at all anachronistic.

Next, lines 1 and 2 ask the student to multiply the 45′ (on the right-hand side of the version in symbols) by 4: *You, 45′ to 4 raise, 3 you see.* To "raise," we remember from page 13, stands for multiplying a concrete magnitude—here the number which represents a composite line segment. The outcome of this multiplication is 3, and the text asks a rhetorical questions: *3, what is that?*

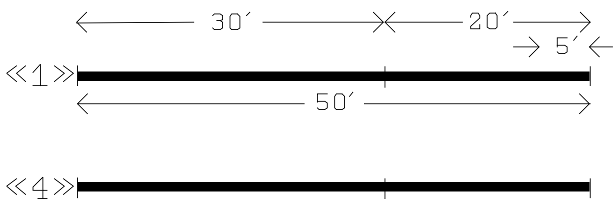

Figure 2.3: Interpretation of TMS XVI, lines 1–3.

The answer to this question is found in lines 2–5. *4 and 1 posit*: First, the student should "posit" 4 and 1. To "posit" means to give a material representation;

here, the numbers should probably be written in the appropriate place in a diagram (Figure 2.3 is a possible interpretation). The number «1» corresponds to the fact that the number $45'$ to the right in the initial equation as well as the magnitudes to the left are all used a single time. The number «4» is "posited" because we are to explain what happens when $45'$ and the corresponding magnitudes are taken 4 times.

50′ and 5′, to tear out, posit: the numbers $50'$ and $5'$ are placed on level «1» of the diagram. This should surprise us: it shows that the student is supposed to know already that the width is $20'$ and the length is $30'$. If he did not, he would not understand that $\ell + w = 50'$ and that $\frac{1}{4}w$ (that which is to be torn out) is $5'$. For the sake of clarity not only the numbers $50'$ and $5'$ but also $30'$ and $20'$ are indicated at level «1» in our diagram even though the text does not speak about them.

Lines 3–5 prove even more convincingly that the student is supposed to know already the solution to the problem (which is thus only a quasi-problem). The aim of the text is thus not to find a solution. As already stated, it is to explain the concepts and procedures that serve to understand and reduce the equation.

These lines explain how and why the initial equation

$$(\ell + w) - \tfrac{1}{4}w = 45'$$

is transformed into

$$4\ell + (4 - 1)w = 3$$

through multiplication by 4.

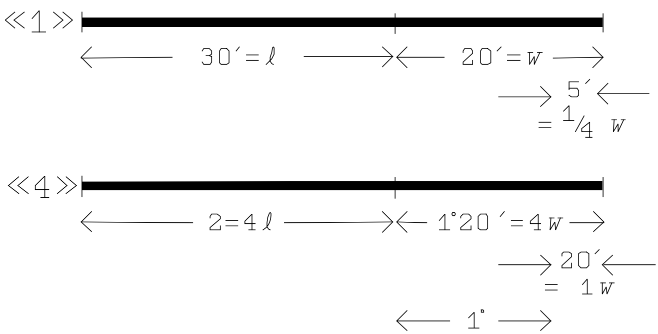

Figure 2.4: Interpretation of TMS XVI, lines 3–5.

This calculation can be followed in Figure 2.4, where the numbers on level «1» are multiplied by 4, giving thereby rise to those of level «4»:

5′ to 4 raise, 1 width: 5′, that is, the $\frac{1}{4}$ of the width, is multiplied by 4, from which results 20′, that is, one width.

20′ to 4 raise, 1°20′ you ⟨see⟩, 4 widths: 20′, that is, 1 width, is multiplied by 4, from which comes 1°20′, thus 4 widths.

30′ to 4 raise, 2 you ⟨see⟩, 4 lengths: 30′, that is 1 length, is multiplied by 4. This gives 2, 4 lengths.

After having multiplied all the numbers of level «1» by 4, and finding thus their counterparts on level «4», the text indicates (lines 4 and 5) what remains when 1 width is eliminated from 4 widths: 20′, *1 width, to tear out, from 1°20′, 4 widths, tear out, 1 you see.*

Finally, the individual constituents of the sum $4\ell + (4-1)\,w$ are identified, as shown in Figure 2.5 *2, the lengths, and 1, 3 widths, heap, 3 you see*: 2, that is, 4 lengths, and 1, that is, $(4-1) = 3$ widths, are added. This gives the number *3*. We have now found the answer to the question of line 2, *3 you see*. *3, what is that?*

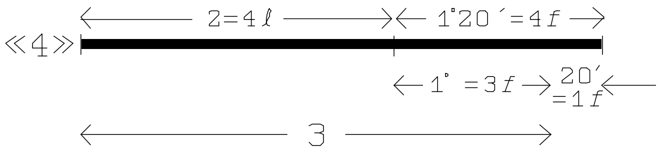

Figure 2.5: Interpretation of TMS XVI, line 5.

But the lesson does not stop here. While lines 1–5 explain how the equation $(\ell+w)-\frac{1}{4}w = 45'$ can be transformed into $4\cdot\ell+(4-1)\cdot w = 3$, what follows in lines 6–10 leads, through division by 4, to a transformation of this equation into

$$1\cdot\ell + \tfrac{3}{4}\cdot w = 45'.$$

For the Babylonians, division by 4 is indeed effectuated as a multiplication by $\frac{1}{4}$. Therefore, line 6 states that $\frac{1}{4} = 15'$: *IGI 4 detach, 15′ you see.* IGI 4 can be found in the table of IGI, that is, of reciprocals (see page 20).

Figure 2.6 shows that this corresponds to a return to level «1»:

15′ to 2, lengths, raise, 30′ you ⟨see⟩, 30′ the length: 2, that is, 4 lengths, when multiplied by $\frac{1}{4}$ gives 30′, that is, 1 length.

15′ to 1 raise, 15′ the contribution of the width. (line 7): 1, that is, 3 widths, is multiplied by $\frac{1}{4}$, which gives 15′, the contribution of the width to the sum 45′. The quantity of widths to which this contribution corresponds is determined in line 8 and 9. In the meantime, the contributions of the length and the width are

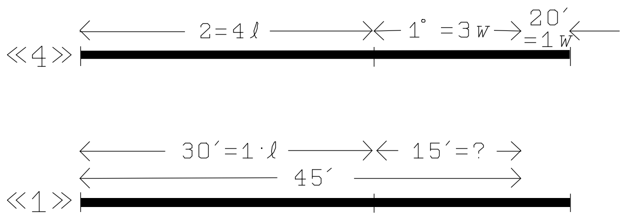

Figure 2.6: Interpretation of TMS XVI, lines 6–12.

memorized: *30′ and 15′ hold*—a shorter expression for *may you head hold*, the formulation used in other texts. We notice the contrast to the material taking note of the numbers 1, 4, 50′ and 5′ by "positing" in the beginning.

The contribution of the width is thus 15′. The end of line 9 indicates that the number of widths to which that corresponds—*the coefficient* of the width, in our language—is $\frac{3}{4}$ (= 45′): *45′ as much as (there is) of widths*. The argument leading to this is of a type known as "simple false position."[4]

Line 8 quotes the statement of the quasi-problem as a justification of what is done (such justifications by quotation are standard): *Since "The 4th of the width, to tear out", it is said to you*. We must therefore find out how much remains of the width when $\frac{1}{4}$ has been removed.

For the sake of convenience, it is "posited" that the quantity of widths is 4 (this is the "false position"). $\frac{1}{4}$ of 4 equals 1 (the text gives this number without calculation). When it is eliminated, 3 remains: *from 4, 1 tear out, 3 you see.*

In order to see to which part of the falsely posited 4 this 3 corresponds, we multiply by $\frac{1}{4}$. Even though this was already said in line 6, it is repeated in line 9 that $\frac{1}{4}$ corresponds to 15′: *IGI 4 de⟨tach⟩, 15′ you see.*

Still in line 9, multiplication by 3 gives the coefficient of the width as 45′ (=$\frac{3}{4}$): *15′ to 3 raise, 45′ you ⟨see⟩, 45′ as much as (there is) of widths.*

Without calculating it line 10 announces that the coefficient of the length is 1. We know indeed from line 1 that a sole length enters into the 45′, without addition nor subtraction. We have thus explained how the equation $4 \cdot \ell + (4 - 1) \cdot w = 3$ is transformed into

$$1 \cdot \ell + \tfrac{3}{4} \cdot w = 45'.$$

[4]"Simple" because there is also a "double false position" that may serve to solve more complex first-degree problems. It consists in making two hypotheses for the solution, which are then "mixed" (as in alloying problems) in such a way that the two errors cancel each other (in modern terms, this is a particular way to make a linear interpolation). Since the Babylonians never made use of this technique, a "false position" always refers to the "simple false position" in what follows.

The end of line 10 presents us with a small riddle: what is the relation between the "true width" and the width which figures in the equations?

The explanation could be the following: a true field might measure 30 [NINDAN] by 20 [NINDAN] (c. 180 m by 120 m, that is, $\frac{1}{3}$ BÙR), but certainly not 30′ by 20′ (3 m by 2 m). On the other hand it would be impossible to draw a field with the dimensions 30×20 in the courtyard of the schoolmaster's house (or any other school; actually, a sandstrewn courtyard is the most plausible support for the diagrams used in teaching). But 30′ by 20′ would fit perfectly (we know from excavated houses), and this order of magnitude is the one that normally appears in mathematical problems. Since there is no difference in writing between 20 and 20′, this is nothing but a possible explanation—but a plausible one, since no alternative seems to be available.

In any case, in line 11 it is found again that the width contributes with 15′, namely by multiplying 20′ (1 width) by the coefficient 45′: *20′ to 45′ raise, 15′ you see.*

In the end, the contribution of the width is eliminated from 45′ (already written ${}_{30}{}^{15}$, that is, as the sum of 30′ and 15′, in agreement with the partition memorized in the end of line 7). 30′ remains, that is, the length: *15′ from ${}^{30}{}_{15'}$ tear out, 30′ you see, 30′ the length.*

All in all, a nice pedagogical explanation, which guides the student by the hand crisscross through the subject "how to transform a first-degree equation, and how to understand what goes on."

Before leaving the text, we may linger on the actors that appear, and which recur in most of those texts that state a problem together with the procedure leading to its solution.[5] Firstly, a "voice" speaking in the first person singular describes the situation which he has established, and formulates the question. Next a different voice addresses the student, giving orders in the imperative or in the second person singular, present tense; this voice cannot be identical with the one that stated the problem, since it often quotes it in the third person, "since he has said."

In a school context, one may imagine that the voice that states the problem is that of the school master, and that the one which addresses the student is an assistant or instructor—"edubba texts,"[6] literary texts about the school and about school life, often refer to an "older brother" whose task it is to give instructions. However, the origin of the scheme appears to be different. Certain texts from the

[5] The present document employs many logograms without phonetic or grammatical complements. Enough is written in syllabic Akkadian, however, to allow us to discern the usual scheme which, in consequence, is imposed upon the translation.

[6] The Sumerian word É.DUB.BA means "tablet house," that is, "school."

early eighteenth century begin “If somebody asks you thus, ‘I have …’.” In these texts the one who asks is a hypothetical person not belonging to the didactical situation—a pretext for a mathematical riddle. The anonymous guide is then the master, originally probably to be identified with a master-surveyor explaining the methods of the trade to his apprentice.

TMS VII #2

This text is rather intricate. Who finds it too opaque may skip it and eventually return to it once familiarized with the Babylonian mode of thought.

17. The fourth of the width to the length I have joined, its seventh
18. until 11 I have gone, over the heap
19. of length and width 5′ it went beyond. You, 4 posit;
20. 7 posit; 11 posit; and 5′ posit.
21. 5′ to 7 raise, 35′ you see.
22. 30′ and 5′ posit. 5′ to 11 raise, 55′ you see.
23. 30′, 20′, and 5′, to tear out, posit. 5′ to 4
24. raise, 20′ you see, 20 the width. 30′ to 4 raise:
25. 2 you see, 2, lengths. 20′ from 20′ tear out.
26. 30′ from 2 tear out, 1°30′ posit, and 5′ to ¿50′, the heap of length and width, join?
27. 7 to 4, of the fourth, raise, 28 you see.
28. 11, the heaps, from 28 tear out, 17 you see.
29. From 4, of the fourth, 1 tear out, 3 you see.
30. IGI 3 detach, 20′ you see. 20′ to 17 raise,
31. 5°40′ you see, 5°40′, (for) the length. 20′ to 5′, the going-beyond, raise,
32. 1′40″ you see, 1′40″, the to-be-joined of the length. 5°40′, (for) the length,
33. from 11, heaps, tear out, 5°20′ you see.
34. 1′40″ to 5′, the going-beyond, join, 6′40″ you see.
35. 6′40″, the to-be-torn-out of the width. 5′, the step,
36. to 5°40′, lengths, raise, 28′20″ you see.
37. 1′40″, the to-be-joined of the length, to 28′20″ join,
38. 30′ you see, 30′ the length. 5′ to 5°20′
39. raise: 26′40″ you see. 6′40″,
40. the to-be-torn-out of the width, from 26′40″ tear out,
41. 20′ you see, 20′ the width.

This is the second, difficult problem from a tablet. The first, easy one (found on page 118 in English translation) can be expressed in symbols in this way:

$$10 \cdot \left(\tfrac{1}{7}[\ell + \tfrac{1}{4}w]\right) = \ell + w.$$

After reduction, this gives the equation

$$\ell \cdot 10 = 6 \cdot (\ell + w).$$

This is an "indeterminate" equation, and has an infinity of solutions. If we have found one of them (ℓ_o, w_o), all the others can be written $(k \cdot \ell_o, k \cdot w_o)$. The text finds one by taking the first factor to the left to be equal to the first factor to the right (thus $\ell = 6$), and the second factor to the right to be equal to the second factor to the right (thus $\ell + w = 10$, whence $w = 4$). Afterwards the solution that has been tacitly aimed at from the beginning is obtained through "raising" to $5'$ (the "step" $\frac{1}{7}[\ell + \frac{1}{4}w]$ that has been "gone" 10 times). Indeed, if $\ell = 6$, $w = 4$, then the "step" is 1; if we want it to be $5'$ (which corresponds to the normal dimensions of a "school rectangle," $\ell = 30'$, $w = 20'$), then the solution must be multiplied by this value. All of this—which is not obvious—is useful for understanding the second problem.

The first problem is "homogeneous"—all its terms are in the first degree in ℓ and w. The second, the one translated above, is inhomogeneous, and can be expressed in symbols in this way:

$$11 \cdot \left(\tfrac{1}{7}[\ell + \tfrac{1}{4}w]\right) = [\ell + w] + 5'.$$

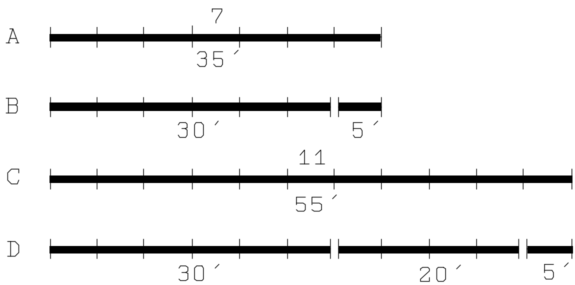

Figure 2.7: Interpretation of TMS XII, lines 21–23.

We take note that $\frac{1}{4}w$ is "joined" to the length; that we take $\frac{1}{7}$ of the outcome; and that afterwards we "go" this segment 11 times. What results "goes beyond" the "heap" of length and width by $5'$. The "heap" is thus no part of what results from the repetition of the step—if it were it could have been "torn out."

The solution begins with a pedagogical explanation in the style of TMS XVI #1, the preceding quasi-problem. Reading well we see that the $5'$ which is "raised" to 7 in line 21 must be the "step" $\frac{1}{7}[\ell + \frac{1}{4}w]$—the raising is a verification that it is really the 7th—and not the "going-beyond" referred to in line 20. Once again the student is supposed to understand that the text is based on the rectangle $\sqsubset\!\sqsupset(30', 20')$. Having this configuration in mind we will be able to follow the explanation of lines 21 to 23 on Figure 2.7: when the "step" $5'$ is "raised" to 7, we get $35'$ (A), which can be decomposed as ℓ and $\frac{1}{4}w$ (B). When it is "raised" to 11 we find $55'$ (C), which can be decomposed as ℓ, w, and $5'$ (D).

Next follows the prescription for solving the equation; is it still formulated in such a way that the solution is supposed to be known. "Raising" to 4 (lines 23 to 25) gives the equivalent of the symbolic equation

$$11 \cdot \left(\tfrac{1}{7}[4\ell + 4 \cdot \tfrac{1}{4}w]\right) = 4 \cdot ([\ell + w] + 5').$$

Not having access to our symbols, the text speaks of $\frac{1}{4}w$ as $5'$, finds that $4 \cdot \frac{1}{4}w$ is equal to $20'$, and identifies that with the width (line 24); then 4ℓ appears as 2, said to represent lengths (line 25).

Now, by means of a ruse which is elegant but not easy to follow, the equation is made homogeneous. The text decomposes $4\ell + w$ as

$$(4-1)\ell - 5' + (w - w) + (\ell + w + 5')$$

and "raises" the whole equation to 7. We may follow the calculation in modern symbolic translation:

$$\begin{aligned} 11 \cdot ([4-1]\ell - 5' + 0 + [\ell + w + 5']) &= (7 \cdot 4) \cdot ([\ell + w] + 5') \\ \Leftrightarrow \quad 11 \cdot ([4-1]\ell - 5') &= (28 - 11) \cdot ([\ell + w] + 5') \\ &= 17 \cdot ([\ell + w] + 5') \\ \Leftrightarrow \quad 11 \cdot \left(\ell - \tfrac{1}{3} \cdot 5'\right) &= \tfrac{1}{3} \cdot 17 \cdot (\ell + w + 5') \\ \Leftrightarrow \quad (\ell - 1'40'') \cdot 11 &= 5°40' \cdot (\ell + w + 5'). \end{aligned}$$

However, the Babylonians did not operate with such equations; they are likely to have inscribed the numbers along the lines of a diagram (see Figure 2.8); that is the reason that the "coefficient" $(4-1)$ does not need to appear before line 29.

As in the first problem of the text, a solution to the homogeneous equation is found by identification of the factors "to the left" with those "to the right" (which is the reason that the factors have been inverted on the left-hand side of the last equation): $\ell - 1'40''$ (now called "the length" and therefore designated

λ in Figure 2.8 thus corresponds to 5°40′, while $\ell + w + 5'$ (referred to as "the heap" of the new length λ and a new width ϕ, that is, $\lambda + \phi$) equals 11; ϕ must therefore be $11 - 5°40' = 5°20'$. Next the text determines the "to-be-joined" (*wāṣbum*) of the length, that is, that which must be joined to the length λ in order to produce the original length ℓ: it equals $1'40''$, since $\lambda = \ell - 1'40''$. Further it finds "the to-be-torn-out" (*nāsḫum*) of the width, that is, that which must be "torn out" from ϕ in order to produce w. Since $\ell + w + 5' = 11$, w must equal $11 - \ell - 5' = 11 - (\lambda + 1'40'') - 5' = (11 - \lambda) - (1'40'' + 5') = \phi - 6'40''$; the "to-be-torn-out" is thus $6'40''$.

But "joining" to λ and "tearing out" from ϕ only gives a *possible* solution, not the one which is intended. In order to have the values for ℓ and w that are aimed at, the step $5'$ is "raised" (as in the first problem) to 5°40′ and 5°20. This gives, respectively, $28'20''$ and $26'40''$; by "joining" to the former its "to-be-joined" and by "tearing out" from the latter its "to-be-torn-out" we finally get $\ell = 30', w = 20'$.

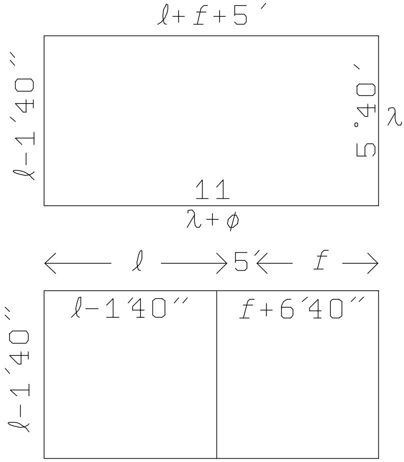

Figure 2.8: The resolution of TMS VII #2.

We must take note of the mastery with which the author avoids to make use in the procedure of his knowledge of the solution (except in the end, where he needs to know the "step" in order to pick the solution that is aimed at among all the possible solutions). The numerical values that are *known* without being *given* serve in the pedagogical explanations; afterwards, their function is to provide *names*—having no symbols like ℓ and λ, the Babylonian needs to use identifications like "the length 30′" and "the length 5′40″" (both are lengths, so the name "length" without any qualifier will not suffice).

Numerical values serve as identifiers in many texts; nonetheless, misunderstandings resulting from a mix-up of *given* and *merely known* numbers are extremely rare.