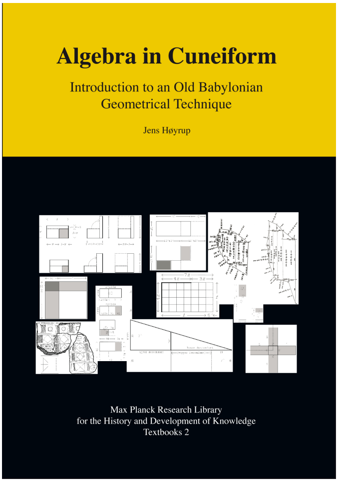
Max Planck Research Library for the History and Development of Knowledge
Textbooks 2

Jens Høyrup:

Introduction: The Issue – and Some Necessary Tools



In: Jens Høyrup: *Algebra in Cuneiform : Introduction to an Old Babylonian Geometrical Technique*
Online version at `http://mprl-series.mpg.de/textbooks/2/`

ISBN 978-3-945561-15-7
First published 2017 by Edition Open Access, Max Planck Institute for the History of Science under

Printed and distributed by:
PRO BUSINESS digital printing Deutschland GmbH, Berlin
`http://www.book-on-demand.de/shop/15336`

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at `http://dnb.d-nb.de`

Chapter 1
Introduction: The Issue – and Some Necessary Tools

"Useless mathematics"

At some moment in the late 1970s, the Danish Union of Mathematics Teachers for the pre-high-school level asked its members a delicate question: to find an application of second-degree equations that fell inside the horizon of their students.

One member did find such an application: the relation between duration and counter numbers on a compact cassette reader (thus an application that at best the parents of today's students will remember!). That was the only answer.

Many students will certainly be astonished to discover that even their teachers do not know why second-degree equations are solved. Students as well as teachers will be no less surprised that such equations have been taught since 1800 BCE without any possible external reference point for the students—actually for the first 2500 years without reference to possible applications *at all* (only around 700 CE did Persian and Arabic astronomers *possibly* start to use them in trigonometric computation).

We shall return to the question why one taught, and still teaches, second-degree equations. But first we shall look at how the earliest second-degree equations, a few first-degree equations and a single cubic equation looked, and examine the way they were solved. We will need to keep in mind that even though some of the problems from which they are derived look practical (they may refer to mercantile questions, to siege ramps and to the division of fields), the mathematical substance is always "pure," that is, deprived of any immediate application outside of mathematics itself.

Rudiments of General History

Mesopotamia ("Land between the rivers") has designated since antiquity the region around the two great rivers Euphrates and Tigris—grossly, contemporary Iraq. Around 3500 BCE, the water level in the Persian Gulf had fallen enough to allow large-scale irrigation agriculture in the southern part of the region, and soon the earliest "civilization" arose, that is, a society centred on towns and organized as a state. The core around which this state took shape was constituted by the great temples and their clergy, and for use in their accounting this clergy invented the earliest script (see the box "Cuneiform Writing," page 10).

The earliest script was purely ideographic (a bit like modern mathematical symbolism, where an expression like $E = mc^2$ can be explained and even pronounced in any language but does not allow us to decide in which language Einstein thought). During the first half of the third millennium, however, phonetic and grammatical complements were introduced, and around 2700 BCE the language was unmistakably Sumerian. From then on, and until c. 2350, the area was divided into a dozen city-states, often at war with each other for water resources. For this reason, the structure of the state was transformed, and the war leader ("king") displaced the temples as the centre of power. From around 2600, a professional specialization emerged, due to wider application of writing. Accounting was no longer the task of the high officials of temple and king: the scribe, a new *profession*, taught in schools and took care of this task.

Around 2340, an Akkadian conqueror subdued the whole of Mesopotamia (Akkadian is a Semitic language, from the same language family as Arabic and Hebrew, and it had been amply present in the region at least since 2600). The Akkadian regional state lasted until c. 2200, after which followed a century of competing city states. Around 2100, the city-state of Ur made itself the centre of a new centralized regional state, whose official language was still Sumerian (even though most of the population, including the kings, probably spoke Akkadian). This "neo-Sumerian" state (known as Ur III) was highly bureaucratized (perhaps more than any other state in history before the arrival of electronic computers), and it seems that the place-value number notation was created in response to the demand of the bureaucracy for convenient calculational instruments (see the box "The Sexagesimal Place-Value System," page 14).

In the long run, the bureaucracy was too costly, and around 2000 a new phase of smaller states begins. After another two centuries another phase of centralization centred around the city of Babylon sets in—from which moment it is meaningful to speak of southern and central Mesopotamia as "Babylonia." By now (but possibly since centuries), Sumerian was definitively dead, and Akkadian had become the principal language—in the south and centre the Babylonian and in the north the Assyrian dialect. None the less, Sumerian survived in the environment of learned scribes—a bit like Latin in Europe—as long as cuneiform writing itself, that is, until the first century CE .

The phase from 2000 until the definitive collapse of the Babylonian central state around 1600 is known as the "Old Babylonian" epoch. All texts analyzed in the following are from its second half, 1800 to 1600 BCE.

The First Algebra and the First Interpretation

Before speaking about algebra, one should in principle know what is meant by that word. For the moment, however, we shall leave aside this question; we shall return to it in the end of the book; all we need to know for the moment is that algebra has to do with equations.

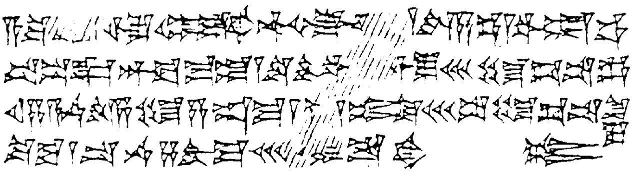


Figure 1.1: The cuneiform version of the problem BM 13901 #1.

Indeed, when historians of mathematics discovered in the late 1920s that certain cuneiform texts (see the box "Cuneiform Writing," page 10) contain "algebraic" problems, they believed everybody knew the meaning of the word.

Let us accept it in order to enter their thinking, and let us look at a very simple example extracted from a text written during the eighteenth century BCE in the transliteration normally used by Assyriologists—as to the function of *italics* and SMALL CAPS, see page 23 and the box "Cuneiform Writing," page 10 (Figure 1.1 shows the cuneiform version of the text):

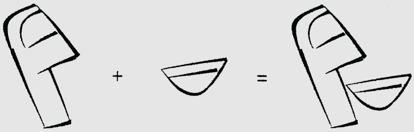
1. A.ŠÀ$^{l[am]}$ *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-E 1 *wa-ṣi-tam*
2. *ta-ša-ka-an ba-ma-at* 1 *te-ḫe-pe* [3]0 *ù* 30 *tu-uš-ta-kal*
3. 15 *a-na* 45 *tu-ṣa-ab-ma* 1-[E] 1 ÍB.SI$_8$ 30 *ša tu-uš-ta-ki-lu*
4. *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

The unprepared reader, finding this complicated, should know that for the pioneers it was almost as complicated. Eighty years later we understand the technical terminology of Old Babylonian mathematical texts; but in 1928 it had not yet been deciphered, and the numbers contained in the texts had to provide the starting point.[1]

[1] However, around 1930 one had to begin with texts that were much more complex than the one we consider here, which was only discovered in 1936. But the principles were the same. The most important contributions in the early years were due to Otto Neugebauer, historian of ancient mathematics and astronomy, and the Assyriologist François Thureau-Dangin.

Cuneiform Writing

From its first beginning, Mesopotamian writing was made on a flattened piece of clay, which was then dried in the air after the inscription (a "tablet"). In the fourth millennium, the signs were drawings made by means of a pointed stylus, mostly drawings of recognizable objects representing simple concepts. Complex concepts could be expressed through combination of the signs; a head and a bowl containing the daily ration of a worker meant "allocation of grain" (and later "to eat").

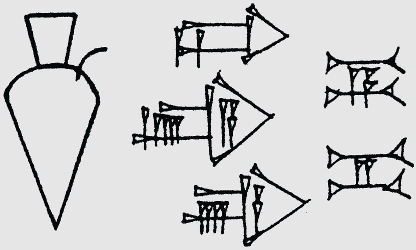


The signs for numbers and measures, however, were made by vertical or oblique impression of a cylindrical stylus.

With time, the character of the script changed in two ways. Firstly, instead of tracing signs consisting of curved lines one impressed them with a stylus with sharp edges, dissolving the curved lines into a sequence of straight segments. In this way, the signs seem to be composed of small wedges (whence the name "cuneiform").

In the second half of the third millennium, numerical and metrological signs came to be written in the same way. The signs became increasingly stylized, loosing their pictographic quality; it is then not possible to guess the underlying drawing unless one knows the historical development behind the sign. Until around 2000 BCE, however, the variations of characters from one scribe to another show that the scribes knew the original drawings.

Let us for instance look at the character which initially depicted a vase with a spout (left).



In the middle we see three third-millennium variants of the same character (because the script was rotated 90 degrees to the left in the second millennium, it is habitual to show the third-millennium script in the same way). If you know the origin, it is still easy to recognize the underlying picture. To the right we see two Old Babylonian variants; here the picture is no longer suggested.

The other change concerns the use of the way the signs were used (which implies that we should better speak of them as "characters"). The Sumerian word for the vase is DUG. As various literary genres developed alongside accounting (for instance, royal inscriptions, contracts and proverb collections), the scribes needed ways to write syllables that serve to indicate grammatical declinations or proper nouns. This syllabic system served also in the writing of Akkadian. For this purpose, signs were used according to their approximate phonetic value; the "vase" may thus stand for the syllables *dug, duk, tug* and *tuk.* In Babylonian writing, the Sumerian sign might also serve as a "logogram" or "word sign" for a word meaning the same as *dug*—namely *karpatum*

Words to be read as logograms or in Sumerian are transliterated in SMALL CAPS; specialists (cf. Appendix B) often distinguish Sumerian words whose phonetic value is supposed to be known, which are then written in s p a c e d w r i t i n g, from those rendered by their "sign name" (corresponding to a *possible* reading), which are written as SMALL CAPS. Phonetic Akkadian writing is transcribed as *italics*.

Assyriologists distinguish "transcriptions" from "transliterations." A "transcription" is an intended translation into Akkadian written in Latin alphabet. In a "transliteration" each cuneiform character is rendered separately according to its presumed phonetic or logographic value.

It was already known that these numbers were written in a place-value system with base 60 but without indication of absolute order of magnitude (see the box "The Sexagesimal System," page 14). We must suppose that the numbers appearing in the text are connected, and that they are of at least approximately the same order of magnitude (we remember that "1" may mean one as well as 60 or $\frac{1}{60}$). Let us therefore try to interpret these numbers in the following order:

$$45'\left(=\tfrac{3}{4}\right)-1^\circ-1^\circ-30'\left(=\tfrac{1}{2}\right)-30'-15'\left(=\tfrac{1}{4}\right)-45'-1^\circ-1^\circ-30'-1^\circ-30'.$$

In order to make the next step one needs some fantasy. Noticing that $30'$ is $\frac{1}{2}\cdot 1$ and $15' = (30')^2$ we may think of the equation

$$x^2 + 1\cdot x = \tfrac{3}{4}.$$

Today we solve it in these steps (neglecting negative numbers, a modern invention):

$$\begin{aligned} x^2+1\cdot x=\tfrac{3}{4} \quad &\Leftrightarrow \quad x^2+1\cdot x+\left(\tfrac{1}{2}\right)^2=\tfrac{3}{4}+\left(\tfrac{1}{2}\right)^2 \\ &\Leftrightarrow \quad x^2+1\cdot x+\left(\tfrac{1}{2}\right)^2=\tfrac{3}{4}+\tfrac{1}{4}=1 \\ &\Leftrightarrow \quad \left(x+\tfrac{1}{2}\right)^2=1 \\ &\Leftrightarrow \quad x+\tfrac{1}{2}=\sqrt{1}=1 \\ &\Leftrightarrow \quad x=1-\tfrac{1}{2}=\tfrac{1}{2}. \end{aligned}$$

As we see, the method is based on addition, to both sides of the equation, of the square on half the coefficient of the first-degree term (x)—here $(\frac{1}{2})^2$. That allows us to rewrite the left-hand side as the square on a binomial:

$$x^2+1\cdot x+\left(\tfrac{1}{2}\right)^2 = x^2+2\cdot\tfrac{1}{2}\cdot x+\left(\tfrac{1}{2}\right)^2=\left(x+\tfrac{1}{2}\right)^2.$$

This small trick is called a "quadratic completion."

Comparing the ancient text and the modern solution we notice that the same numbers occur in almost the same order. The same holds for many other texts. In the early 1930s historians of mathematics thus became convinced that between 1800 and 1600 BCE the Babylonian scribes knew something very similar to our equation algebra. This period constitutes the second half of what is known as the "Old Babylonian" epoch (see the box "Rudiments of General History," page 7).

The next step was to interpret the texts precisely. To some extent, the general, non-technical meaning of the vocabulary could assist. In line 1 of the problem on page 9, *ak-mur* may be translated "I have heaped." An understanding of the "heaping" of two numbers as an addition seems natural and agrees with the observation that the "heaping" of $45'$ and $15'$ (that is, of $\frac{3}{4}$ and $\frac{1}{4}$) produces 1. When other texts "raise" (*našûm*) one magnitude to another one, it becomes more difficult. However, one may observe that the "raising" of 3 to 4 produces 12, while 5 "raised" to 6 yields 30, and thereby guess that "raising" is a multiplication.

In this way, the scholars of the 1930s came to choose a purely arithmetical interpretation of the operations—that is, as additions, subtractions, multiplications and divisions *of numbers*. This translation offers an example:[2]

1. I have added the surface and (the side of) my square: 45′.
2. You posit 1°, the unit. You break into two 1° : 30′. You multiply (with each other) [30′] and 30′:
3. 15′. You join 15′ to 45′: 1°. 1° is the square of 1°. 30′, which you have multiplied (by itself),
4. from 1° you subtract: 30′ is the (side of the) square.

Such translations are still found today in general histories of mathematics. They explain the numbers that occur in the texts, and they give an almost modern impression of the Old Babylonian methods. There is no fundamental difference between the above translation and the solution by means of equations. If the side of the square is x, then its area is x^2. Therefore, the first line of the text—the problem to be solved—corresponds to the equation $x^2 + 1 \cdot x = \frac{3}{4}$. Continuing the reading of the translation we see that it follows the symbolic transformations on page 12 step by step.

However, even though the present translation as well as others made according to the same principles explain the numbers of the texts, they agree less well with their words, and sometimes not with the order of operations. Firstly, these translations do not take the geometrical character of the terminology into account, supposing that words and expressions like "(the side of) my square," "length," "width" and "area" of a rectangle denote nothing but unknown numbers and their products. It must be recognized that in the 1930s that did not seem impossible *a priori*—we too speak of 3^2 as the "square of 3" without thinking of a quadrangle.

But there are other problems. The most severe is probably that the number of operations is too large. For example, there are two operations that in the traditional interpretation are understood as addition: "to join to" (*waṣābum*/DAḪ, the infinitive corresponding to the *tu-ṣa-ab* of our text) and "to heap" (*kamārum*/GAR.GAR, from which the *ak-mur* of the text). Both operations are thus found in our brief text, "heaping" in line 1 (where it appears as "add") and "joining" in line 3.

[2] A literal retranslation of François Thureau-Dangin's French translation. Otto Neugebauer's German translation is equivalent except on one point: where Thureau-Dangin translated "1°, the unit" Neugebauer proposed "1, the coefficient." He also transcribed place-value numbers differently.

The Sexagesimal Place-Value System

The Old Babylonian mathematical texts make use of a place-value number system with base 60 with no indication of a "sexagesimal point." In our notation, which also employs place value, the digit "1" may certainly represent the number 1, but also the numbers $10, 100, \ldots$, as well as $0.1, 0.01, \ldots$. Its value is determined by its distance from the decimal point.

Similarly, "45" written by a Babylonian scribe may mean 45; but it may also stand for $\frac{45}{60}$ (thus $\frac{3}{4}$); for $45 \cdot 60$; etc. No decimal point determines its "true" value. The system corresponds to the slide rule of which engineers made use before the arrival of the electronic pocket calculator. This device also had no decimal point, and thus did not indicate the absolute order of magnitude. In order to know whether a specific construction would ask for $3.5m^3$, $35m^3$ or $350m^3$ of concrete, the engineer had recourse to mental calculation.

For writing numbers between 1 and 59, the Babylonians made use of a vertical wedge (𒁹) repeated until 9 times in fixed patterns for the numbers 1 to 9, and of a *Winkelhaken* (a German loanword originally meaning "angular hook") (𒌋) repeated until 5 times for the numbers $10, 20, \ldots, 50$.

A modern reader is not accustomed to reading numbers with undetermined order of magnitude. In translations of Babylonian mathematical texts it is therefore customary to indicate the order of magnitude that has to be attributed to numbers. Several methods to do that are in use. In the present work we shall employ a generalization of the degree-minute-second notation. If 𒌋𒐊 means $\frac{15}{60}$, we shall transcribe it $15'$, if it corresponds to $\frac{15}{60 \cdot 60}$, we shall write $15''$. If it represents $15 \cdot 60$, we write 15`, etc. If it stands for 15, we write 15 or, if that is needed in order to avoid misunderstandings, 15°. 𒌋𒐊 understood as $10+5 \cdot 60^{-1}$ will thus be transcribed 10°5′

𒌍 understood as 30′ thus means $\frac{1}{2}$.
𒐏𒐊 understood as 45′ means $\frac{3}{4}$.
𒌋𒈫 understood as 12′ means $\frac{1}{5}$; understood as 12` it means 720.
𒌋 understood as 10′ means $\frac{1}{6}$.
𒌋𒐋𒐏 may mean 16`40 = 1000 or 16°40′ = $16\frac{2}{3}$, etc.
𒁹𒐏 may mean 1`40 = 100, 1°40′ = $1\frac{2}{3}$, 1′40″ = $\frac{1}{36}$, etc.

Outside school, the Babylonians employed the place-value system exclusively for intermediate calculations (exactly as an engineer used the slide rule fifty years ago). When a result was to be inserted into a contract or an account, they could obviously not allow themselves to be ambiguous; other notations allowed them to express the precise number they intended.

Certainly, we too know about synonyms even within mathematics—for instance, "and," "added to" and "plus"; the choice of one word or the other depends on style, on personal habits, on our expectations concerning the interlocutor, and so forth. Thureau-Dangin , as we see, makes use of them, following the distinctions of the text by speaking first of "addition" and second of "joining"; but he argues that there is no conceptual difference, and that nothing but synonyms are involved—"there *is* only one multiplication," as he explains without noticing that the argument is circular.

Synonyms, it is true, can also be found in Old Babylonian mathematics. Thus, the verbs "to tear out" (*nasāḫum*/ZI) and "to cut off" (*ḫarāṣum*/KUD) are names for the same subtractive operation: they can be used in strictly analogous situations. The difference between "joining" and "heaping," however, is of a different kind. No text exists which refers to a quadratic completion (above, page 12) as a "heaping." "Heaping," on the other hand, is the operation to be used when an area and a linear extension are added. These are thus distinct operations, not two different names for the same operation. In the same way, there are two distinct "subtractions," four "multiplications" and even two different "halves." We shall come back to this.

A translation which mixes up operations which the Babylonians treated as distinct may explain why the Babylonian calculations lead to correct results; but it cannot penetrate their mathematical thought.

Further, the traditional translations had to skip certain words which seemed to make no sense. For instance, a more literal translation of the last line of our small problem would begin "from the inside of 1°" (or even "from the heart" or "from the bowels"). Not seeing how a number 1 could possess an "inside" or "bowels," the translators tacitly left out the word.

Other words were translated in a way that differs so strongly from their normal meaning that it must arouse suspicion. Normally, the word translated "unity" by Thureau-Dangin and "coefficient" by Neugebauer (*waṣītum*, from *waṣûm*, "to go out") refers to something that sticks out, as that part of a building which architects speak about as a "projection." That must have appeared absurd—how can a number 1 "stick out"? Therefore the translators preferred to make the word correspond to something known in the mathematics of their own days.

Finally, the order in which operations are performed is sometimes different from what seems natural in the arithmetical reading.

In spite of these objections, the interpretation that resulted in the 1930s was an impressive accomplishment, and it remains an excellent "first approximation." The scholars who produced it pretended nothing more. Others however, not least historians of mathematics and historically interested mathematicians, took it to be the unique and final decipherment of "Babylonian algebra"—so impressive were the results that were obtained, and so scary the perspective of being forced to read the texts in their original language. Until the 1980s, nobody noticed that certain apparent synonyms represent distinct operations.[3]

A New Reading

As we have just seen, the arithmetical interpretation is unable to account for the words which the Babylonians used to describe their procedures. Firstly, it conflates operations that the Babylonians treated as distinct; secondly, it is based on operations whose order does not always correspond to that of the Babylonian calculations. Strictly speaking, rather than an interpretation it thus represents a control of the correctness of the Babylonian methods based on modern techniques.

A genuine interpretation —a reading of what the Old Babylonian calculators thought and did—must take two things into account: on one hand, the results obtained by the scholars of the 1930s in their "first approximation"; on the other, the levels of the texts which these scholars had to neglect in order to create this first approximation.

In the following chapters we are going to analyze a number of problems in a translation that corresponds to such an interpretation. First some general information will be adequate.

Representation and "variables"

In our algebra we use x and y as substitutes or names for unknown *numbers*. We use this algebra as a tool for solving problems that concern other kinds of magnitudes, such as prices, distances, energy densities, etc.; but in all such cases we consider these other quantities as represented by numbers. For us, numbers constitute the *fundamental representation*.

With the Babylonians, the fundamental representation was geometric. Most of their "algebraic" problems concern rectangles with length, width and area[4], or

[3]Nobody, except perhaps Neugebauer, who on one occasion observes (correctly) that a text makes use of a wrong multiplication. In any case it must be noticed that neither he nor Thureau-Dangin ever chooses a wrong operation when restituting the missing part of a broken text.

[4]More precisely, the word translated "length" signifies "distance"/"extension"/"length" while that which is translated "width" means "front"/"forehead"/"head." They refer to the idea of a long and narrow irrigated field. The word for the area (*eqlum*/A.ŠÀ) originally means "field" but in order to reserve it for technical use the texts use other (less adequate) words when speaking of genuine fields to be divided. In what follows, the term will be translated "surface," which has undergone a similar shift of meaning, and which stands both for the spatial entity and its area.

A similar distinction is created by other means for lengths and widths. If these stand for "algebraic" variables they are invariably written with the logograms UŠ and SAĜ; if used for general purposes (the length of a wall, a walking distance) they may be provided with phonetic complements or written syllabically as *šiddum* and *pūtum*.

squares with side and area. We shall certainly encounter a problem below (YBC 6967, page 46) that asks about two unknown *numbers*, but since their product is spoken of as a "surface" it is evident that these numbers are *represented* by the sides of a rectangle.

An important characteristic of Babylonian geometry allows it to serve as an "algebraic" representation: it always deals with *measured* quantities. The measure of its segments and areas may be treated as *unknown*—but even then it exists as a numerical measure, and the problem consists in finding its value.

Units

Every measuring operation presupposes a metrology, a system of measuring units; the numbers that result from it are concrete numbers. That cannot be seen directly in the problem that was quoted above on page 9; mostly, the mathematical texts do not show it since they make use of the place-value system (except, occasionally, when given magnitudes or final results are stated). In this system, all quantities of the same kind were measured in a "standard unit" which, with very few exceptions, was not stated but tacitly understood.

The standard unit for *horizontal distance* was the NINDAN, a "rod" of c. 6 m.[5] In our problem, the side of the square is thus $\frac{1}{2}$ NINDAN, that is, c. 3 m. For *vertical distances* (heights and depths), the basic unit was the KÙŠ, a "cubit" of $\frac{1}{12}$ NINDAN (that is, c. 50 cm).

The standard unit for *areas* was the SAR, equal to 1 NINDAN^2. The standard unit for volumes had the same name: the underlying idea was that a base of 1 NINDAN^2 was provided with a standard thickness of 1 KÙŠ. In agricultural administration, a better suited area unit was used, the BÙR, equal to 30` SAR, c. $6\frac{1}{2}$ ha.

The standard unit for *hollow measures* (used for products conserved in vases and jars, such as grain and oil) was the SÌLA, slightly less than one litre. In practical life, larger units were often used: 1 BÁN = 10 SÌLA, 1 PI = 1` SÌLA, and 1 GUR, a "tun" of 5` SÌLA.

Finally, the standard unit for *weights* was the shekel, c. 8 gram. Larger units were the *mina* , equal to 1` shekel (thus close to a pound)[6] and the GÚ, "a load"

[5] In the absence of a sexagesimal point it is in principle impossible to know whether the basic unit was 1 NINDAN, 60 NINDAN or $\frac{1}{60}$ NINDAN. The choice of 1 NINDAN represents what (for us, at least) seems most natural for an Old Babylonian calculator, since it already exists as a unit (which is also true for 60 NINDAN but not for $\frac{1}{60}$ NINDAN) and because distances measured in NINDAN had been written without explicit reference to the unit for centuries before the introduction of the place-value system.

[6] It is not to be excluded that the Babylonians thought of the *mina* as standard unit, or that they kept both possibilities open.

equal to 1`` shekel, c. 30 kilogram. This last unit is equal to the talent of the Bible (where a talent *of silver* is to be understood).

Additive Operations

There are two additive operations. One (*kamārum*/UL.GAR/GAR.GAR), as we have already seen, can be translated "to heap *a* and *b*," the other (*waṣābum*/DAḪ) "to join *j* to *S*." "Joining" is a concrete operation which conserves the identity of *S*. In order to understand what that means we may think of "my" bank deposit *S*; adding the interest *j* (in Babylonian called precisely *ṣibtum*, "the joined," a noun derived from the verb *waṣābum*) does not change its identity as *my* deposit. If a geometric operation "joins" *j* to *S*, *S* invariably remains in place, whereas, if necessary, *j* is moved around.

"Heaping," to the contrary, may designate the addition of abstract numbers. Nothing therefore prevents from "heaping" (the number measuring) an area and (the number measuring) a length. However, even "heaping" often concerns entities allowing a concrete operation.

The sum resulting from a "joining" operation has no particular name; indeed, the operation creates nothing new. In a heaping process, on the other hand, where the two addends are absorbed into the sum, this sum has a name (*nakmartum*, derived from *kamārum*, "to heap") which we may translate "the heap"; in a text where the two constituents remain distinct, a plural is used (*kimrātum*, equally derived from *kamārum*); we may translate it "the things heaped" (AO 8862 #2, translated in Chapter 4, page 60).

Subtractive Operations

There are also two subtractive operations. One (*nasāḫum*/ZI), "from *B* to tear out *a*" is the inverse of "joining"; it is a concrete operation which presupposes *a* to be a constituent part of *B*. The other is a comparison, which can be expressed "*A* over *B*, *d* goes beyond" (a clumsy phrase, which however maps the structure of the Babylonian locution precisely). Even this is a concrete operation, used to compare magnitudes of which the smaller is not part of the larger. At times, stylistic and similar reasons call for the comparison being made the other way around, as an observation of *B* falling short of *A* (note 4, page 48 discusses an example).

The difference in the first subtraction is called "the remainder" (*šapiltum*, more literally "the diminished"). In the second, the excess is referred to as the "going-beyond" (*watartum*/DIRIG).

There are several synonyms or near-synonyms for "tearing out." We shall encounter "cutting off" (*ḫarāṣum*) (AO 8862 #2, page 60) and "make go away" (*šutbûm*) (VAT 7532, page 65).

“Multiplications”

Four distinct operations have traditionally been interpreted as multiplication.

First, there is the one which appears in the Old Babylonian version of the multiplication table. The Sumerian term (A.RÁ, derived from the Sumerian verb RÁ, “to go”) can be translated “steps of.” For example, the table of the multiples of 6 runs:

1 step of 6 is 6
2 steps of 6 are 12
3 steps of 6 are 18
…

Three of the texts we are to encounter below (TMS VII #2, page 34, TMS IX #3, page 57, and TMS VIII #1, page 78) also use the Akkadian verb for “going” (*alākum*) to designate the repetition of an operation: the former two repeat a magnitude s n times, with outcome $n \cdot s$ (TMS VII #2, line 18; TMS IX #3, line 21); TMS VIII #1 line 1 joins a magnitude s n times to another magnitude A, with outcome $A + n \cdot s$.

The second “multiplication” is defined by the verb “to raise” (*našûm*/ÍL/NIM). The term appears to have been used first for the calculation of volumes: in order to determine the volume of a prism with a base of G SAR and a height of h KÙŠ, one “raises” the base with its standard thickness of 1 KÙŠ to the real height h. Later, the term was adopted by analogy for all determinations of a concrete magnitude by multiplication. “Steps of” instead designates the multiplication of an abstract number by another abstract number.

The third “multiplication” (*šutakūlum*/GU_7.GU_7), “to make p and q hold each other”—or simply, because that is almost certainly what the Babylonians thought of, “make p and q hold (namely, hold a rectangle)”[7]—is no real multiplication. It always concerns two line segments p and q, and “to make p and q hold” means to construct a rectangle contained by the sides p and q. Since p and q as well as the area A of the rectangle are all measurable, almost all texts give the numerical value of A immediately after prescribing the operation—“make 5 and 5 hold: 25”—without mentioning the numerical multiplication of 5 by 5 explicitly. But there are texts that speak separately about the numerical multiplication, as “p steps of q,” after prescribing the construction, or which indicate that the process of “making hold” creates “a surface”; both possibilities are exemplified in AO 8862 #2 (page 60). If a rectangle exists already, its area is determined by “raising,” just as the area of a triangle or a trapezium. Henceforth we shall designate the rectangle which is “held” by the segments p and q by the symbol $\sqsubset\!\sqsupset(p,q)$,

[7] The verbal form used would normally be causative-reciprocative. However, at times the phrase used is “make p together with q hold” which seems to exclude the reciprocative interpretation.

while □(a) will stand for the square which a segment a "holds together with itself" (in both cases, the symbol designates the configuration as well the area it contains, in agreement with the ambiguity inherent in the concept of "surface"). The corresponding numerical multiplications will be written symbolically as $p \times q$ and $a \times a$.

The last "multiplication" (*eṣēpum*) is also no proper numerical multiplication. "To repeat" or "to repeat until n" (where n is an integer small enough to be easily imagined, at most 9) stands for a "physical" doubling or n-doubling—for example that doubling of a right triangle with sides (containing the right angle) a and b which produces a rectangle ⊏⊐(a, b).

Division

The problem "what should I raise to d in order to get P?" is a *division problem*, with answer $P \div d$. Obviously, the Old Babylonian calculators knew such problems perfectly well. They encountered them in their "algebra" (we shall see many examples below) but also in practical planning: a worker can dig N NINDAN irrigation canal in a day; how many workers will be needed for the digging of 30 NINDAN in 4 days? In this example the problem even occurs twice, the answer being $(30 \div 4) \div N$. But division was no separate *operation* for them, only a problem type.

In order to divide 30 by 4, they first used a table (see Figure 1.2), in which they could read (but they had probably learned it by heart in school[8]) that IGI 4 is 15′; afterwards they "raised" 15′ to 30 (even for that tables existed, learned by heart at school), finding 7°30′.[9]

Primarily, IGI n stands for *the reciprocal of n as listed in the table* or at least as easily found from it, not the number $\frac{1}{n}$ abstractly. In this way, the Babylonians solved the problem $P \div d$ via a multiplication $P \cdot \frac{1}{d}$ *to the extent that this was possible*.

[8]When speaking of a "school" in the Old Babylonian context we should be aware that we only know it from textual evidence. No schoolroom has been identified by archaeologists (what was once believed to be school rooms has turned out to be for instance store rooms). We therefore do not know whether the scribes were taught in palace or temple schools or in the private homes of a master scribe instructing a handful of students; most likely, many were taught by private masters. The great number of quasi-identical copies of the table of reciprocals that were prepared in order to be learned by heart show, however, that future scribes were not (or not solely) taught as apprentices of a working scribe but according to a precisely defined curriculum; this is also shown by other sources.

[9]It may seem strange that the multiplication of IGI 4 by 30 is done by "raising." Is this not a multiplication of a number by a number? Not necessarily, according the expression used in the texts when IGI 4 has to be found: they "detach" it. The idea is thus a splitting into 4 equal parts, one of which is detached. It seems that what was originally split (when the place-value system was constructed) was a length—namely 1` [NINDAN], not 1 [NINDAN]. This Ur-III understanding had certainly been left behind; but the terminological habit had survived.

Of 1, its 2/3	40	27, its IGI	2 13 20
Its half	30	30, its IGI	2
3, its IGI	20	32, its IGI	1 52 30
4, its IGI	15	36, its IGI	1 40
5, its IGI	12	40, its IGI	1 30
6, its IGI	10	45, its IGI	1 20
8, its IGI	7 30	48, its IGI	1 15
9, its IGI	6 40	50, its IGI	1 12
10, its IGI	6	54, its IGI	1 6 40
12, its IGI	5	1, its IGI	1
15, its IGI	4	1 4, its IGI	56 15
16, its IGI	3 45	1 12, its IGI	50
18, its IGI	3 20	1 15, its IGI	48
20, its IGI	3	1 20, its IGI	45
24, its IGI	2 30	1 21, its IGI	44 26 40
25, its IGI	2 24		

Figure 1.2: Translation of the Old Babylonian table of reciprocals (IGI).

However, this was only possible if n appeared in the IGI table. Firstly, that required that n was a "regular number," that is, that $\frac{1}{n}$ could be written as a finite "sexagesimal fraction."[10] However, of the infinitely many such numbers only a small selection found place in the table—around 30 in total (often, 1 12, 1 15 and 1 20 are omitted "to the left" since they are already present "to the right").

In practical computation, that was generally enough. It was indeed presupposed that all technical constants—for example, the quantity of dirt a worker could dig out in a day—were simple regular numbers. The solution of "algebraic" problems, on the other hand, often leads to divisions by a non-regular divisor d. In such cases, the texts write "what shall I posit to d which gives me A?", giving immediately the answer "posit Q, A will it give you."[11] That has a very natural explanation: these problems were constructed backwards, from known results. Divisors would therefore always divide, and the teacher who constructed a problem already knew the answer as well as the outcome of divisions leading to it.

[10] And, tacitly understood, that n itself can be written in this way. It is not difficult to show that all "regular numbers" can be written $2^p \cdot 3^q \cdot 5^r$, where p, q and r are positive or negative integers or zero. 2, 3 and 5 are indeed the only prime numbers that divide 60. Similarly, the "regular numbers" in our decimal system are those that can be written $2^p \cdot 5^q$, 2 and 5 being the only prime divisors of 10.

[11] The expression "posit to" refers to the way simple multiplication exercises were written in school: the two factors were written one above the other (the second being "posited to" the first), and the result below both.

Halves

$\frac{1}{2}$ may be a fraction like any other: $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. This kind of half, if it is the half of something, is found by raising that thing to 30′. Similarly, its $\frac{1}{3}$ is found by raising to 20′, etc. This kind of half we shall meet in AO 8862 #2 (page 60).

But $\frac{1}{2}$ (in this case necessarily the half of something) may also be a "natural" or "necessary" half, that is, a half that could be nothing else. The radius of a circle is thus the "natural" half of the diameter: no other part could have the same role. Similarly, it is by necessity the exact half of the base that must be raised to the height of a triangle in order to give the area—as can be seen on the figure used to prove the formula (see Figure 1.3).

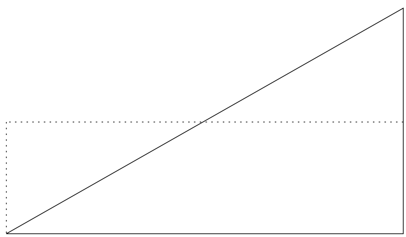


Figure 1.3

This "natural" half had a particular name (*bāmtum*), which we may translate "moiety." The operation that produced it was expressed by the verb "to break" (*ḫepûm*/GAZ)—that is, to bisect, to break in two equal parts. This meaning of the word belongs specifically to the mathematical vocabulary; in general usage the word means to crush or break in any way (etc.).

Square and "square root"

The product $a \cdot a$ played no particular role, neither when resulting from a "raising" nor from an operation of "steps of." A square, in order to be something special, had to be a geometric square.

But the geometric square did have a particular status. One might certainly "make *a* and *a* hold" or "make *a* together with itself hold"; but one might also "make *a* confront itself" (*šutamḫurum*, from *maḫārum* "to accept/ receive/ approach/ welcome"). The square seen as a geometric configuration was a "confrontation" (*mitḫartum*, from the same verb).[12] Numerically, its value was identified with the length of the side. A Babylonian "confrontation" thus *is* its side

[12]More precisely, the Babylonian word stands for "a situation characterized by the confrontation of equals."

while it *has* an area; inversely, our square (identified with what is contained and not with the frame) *is* an area and *has* a side. When the value of a "confrontation" (understood thus as its side) is found, another side which it meets in a corner may be spoken of as its "counterpart"—*meḫrum* (similarly from *maḫārum*), used also for instance about the exact copy of a tablet.

In order to say that *s* is the side of a square area *Q*, a Sumerian phrase (used already in tables of inverse squares probably going back to Ur III, see imminently) was used: "by *Q, s* is equal"—the Sumerian verb being ÍB.SI$_8$. Sometimes, the word ÍB.SI$_8$ is used as a noun, in which case it will be translated "the equal" in the following. In the arithmetical interpretation, "the equal" becomes the square root.

Just as there were tables of multiplication and of reciprocals, there were also tables of squares and of "equals." They used the phrases "*n* steps of *n*, n^2" and "by n^2, *n* is equal" ($1 \leq n \leq 60$). The resolution of "algebraic" problems, however, often involves finding the "equals" of numbers which are not listed in the tables. The Babylonians did possess a technique for finding approximate square roots of non-square numbers—but these *were* approximate. The texts instead give the exact value, and once again they can do so because the authors had constructed the problem backward and therefore knew the solution. Several texts, indeed, commit calculational errors, but in the end they give the square root of the number that *should* have been calculated, not of the number actually resulting! An example of this is mentioned in footnote 8, page 73.

Concerning the Texts and the Translations

The texts that are presented and explained in the following are written in Babylonian, the language that was spoken in Babylonia during the Old Babylonian epoch. Basically they are formulated in syllabic (thus phonetic) writing—that which appears as *italics* on page 11. All also make use of logograms that represent a whole word but indicate neither the grammatical form nor the pronunciation (although grammatical complements are sometimes added to them); these logograms are transcribed in SMALL CAPS (see the box "Cuneiform Writing," page 10). With rare exceptions, these logograms are borrowed from Sumerian, once the main language of the region and conserved as a scholars' language until the first century CE (as Latin in Europe until recently). Some of these logograms correspond to technical expressions already used as such by the Sumerian scribes; IGI is an example. Others serve as abbreviations for Babylonian words, more or less as *viz* in English, which represents the shorthand for *videlicet* in medieval Latin manuscripts but is pronounced *namely*.

As already indicated, our texts come from the second half of the Old Babylonian epoch, as can be seen from the handwriting and the language. Unfortunately it is often impossible to say more, since almost all of them come from illegal diggings and have been bought by museums on the antiquity market in Baghdad or Europe.

We have no direct information about the authors of the texts. They never present themselves, and no other source speaks of them. Since they knew how to write (and more than the rudimentary syllabic of certain laymen), they must have belonged to the broad category of scribes; since they knew how to calculate, we may speak about them as "calculators"; and since the format of the texts refers to a didactical situation, we may reasonably assume that they were school teachers.[13]

All this, however, results from indirect arguments. Plausibly, the majority of scribes never produced mathematics on their own beyond simple computation; few were probably ever trained at the high mathematical level presented by our texts. It is even likely that only a minority of school teachers *taught* such matters. In consequence, and because several voices speak through the texts (see page 33), it is often preferable to pretend that it is the text itself which "gives," "finds," "calculates," etc.

The English translations that follow—all due to the author of the book—do not distinguish between syllabically and logographically written words (readers who want to know must consult the transliterations in Appendix B). Apart from that, they are "conformal"—that is, they are faithful to the original, in the structure of phrases[14] as well as by using always distinct translations for words that are different in the original and the same translation for the same word every time it occurs unless it is used in clearly distinct functions (see the list of "standard translations" on page 129). In as far as possible the translations respect the non-technical meanings of the Babylonian words (for instance "breaking" instead of "bisecting") and the relation between terms (thus "confront itself" and "confrontation"—while "counterpart" had to be chosen unrelated of the verbal root in order to respect the use of the same word for the copy of a tablet).

This is not to say that the Babylonians did not have a technical terminology but only their everyday language; but it is important that the technical meaning of a word be learned from its uses within the Old Babylonian texts and not borrowed

[13] On the problem of the "school" see note 8, page 20, and page 101.

[14] In Akkadian, the verb comes in the end of the phrase. This structure allows a number to be written a single time, first as the outcome of one calculation and next as the object of another one. In order to conserve this architecture of the text ("number(s)/operation: resulting number/new operation"), this final position of the verb is respected in the translations, ungrammatical though it is. The reader will need to get accustomed (but non-English readers should not learn it so well as to use the construction independently!).

(with the risk of being badly borrowed, as has often happened) from our modern terminology.

The Babylonian language structure is rather different from that of English, for which reason the conformal translations are far from elegant. But the principle of conformality has the added advantage that readers who want to can follow the original line for line in Appendix B (the bibliographic note on page 149 indicates where the few texts not rendered in the appendix were published).

In order to avoid completely illegible translations, the principle is not followed to extremes. In English one has to choose whether a noun is preceded by a definite or an indefinite article; in Babylonian, as in Latin and Russian, that is not the case. Similarly, there is no punctuation in the Old Babylonian texts (except line breaks and a particle that will be rendered ":"), and the absolute order of magnitude of place-value numbers is not indicated; minimal punctuation as well as indications of order of magnitude (′,‵ and °) have been added. Numbers that are written in the original by means of numerals have been translated as Arabic numerals, while numbers written by words (including logograms) have been translated as words; mixed writings appear mixed (for instance, "the 17th" and even "the 3rd" for the third).

Inscribed clay survives better than paper—particularly well when the city burns together with its libraries and archives, but also when discarded as garbage. None the less, almost all the tablets used for what follows are damaged. On the other hand, the language of the mathematical texts is extremely uniform and repetitive, and therefore it is often possible to reconstruct damaged passages from parallel passages on the same tablet. In order to facilitate reading the reconstructions are only indicated in the translations (as ¿…?) if their exact words are not completely certain. Sometimes a scribe has left out a sign, a word or a passage when writing a tablet which however can be restored from parallel passages on the same or closely kindred tablets. In such cases the restitution appears as ⟨…⟩, while {…} stands for repetitions and other signs written by error (the original editions of the texts give the complete information about destroyed and illegible passages and scribal mistakes). Explanatory words inserted into the texts appear within rounded brackets (…).

Clay tablets have names, most often museum numbers. The small problem quoted above is the first one on the tablet BM 13901—that is, tablet #13901 in the British Museum tablet collection. Other names begin AO (Ancient Orient, Louvre, Paris), VAT (Vorderasiatische Texte, Berlin) or YBC (Yale Babylonian texts). TMS refers to the edition *Textes mathématiques de Suse* of a Louvre collection of tablets from Susa, an Iranian site in the eastern neighborhood of Babylon.

The tablets are mostly inscribed on both surfaces ("obverse" and "reverse"), sometimes in several columns, sometimes also on the edge; the texts are divided in lines read from left to right. Following the original editions, the translations indicate line numbers and, if actual, obverse/reverse and column.